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Interval Arithmetic for Input-output Models with Inexact Data

MAX E. JERRELL
Department of Business Administration, Northern Arizona University, Flagstaff, AZ 86011-4710, U.S.A.

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Abstract. Input-output models are subject to uncertainty. If these models are solved without regard to the effects of the uncertainty the solutions can be substantially in error. Interval arithmetic offers a means by which the effects of this uncertainty can be assessed. They also offer a means of evaluating changes in the technical coefficients and a means of determining inverse important coefficients.

Key words: input-output models, uncertainty, interval arithmetic.

1. Introduction

The values of the technical coefficients of input-output models are usually not precisely known. The effects of uncertainty are rarely examined though they can be substantial. Kelly and Wyckoff (1989) report that the estimate of GNP for 1983 was revised by \$100 billion when the 1977 national input-output benchmark was released. The problem of uncertainty is particularly true for regional models where analysts rarely have the resources to conduct samples and must adapt national tables to regional economies. West (1986) has derived a method of producing customary confidence limits for multipliers in terms of the parameters of the distributions of the technical coefficients. This requires some way of determining the distributions of the coefficients and of estimating their parameters. This may be a practical impossibility, particularly for regional economists. Interval arithmetic offers a promising method of evaluating the effects of uncertainty when subjective uncertainty assignments can be made to the technical coefficients. In what follows, we will consider a simple input-output model

$$x = Ax + b$$

$$(I - A)x - Lx - b$$

where A is a matrix of technical coefficients, b a vector of final demand, and x is the level of output from various industries. It will be useful for what follows later to designate the Leontief matrix as $L = (I - A)$.

The rules of interval arithmetic that are needed for this research are discussed in Section 2. Section 3 considers various solutions to interval equations. M-matrices, a special class of matrices, are discussed in Section 4 and example is shown in Section 5. Matrices that are not M-matrices are considered in Section 6 and the use of interval arithmetic to discover inverse important coefficients is discussed in Section 7. A brief conclusion is in Section 8.

2. Interval Arithmetic

Interval arithmetic was developed by R. E. Moore (1959) while studying the propagation and control of truncation and roundoff error using floating point arithmetic on a digital computer. Moore (1979) was able to generalize this work into an arithmetic independent of machine considerations.

One of the nice features of interval arithmetic is that the rules are easy to implement. In what follows upper case letters, such as X, will be used to indicate intervals while lower case letters, such as x, will indicate point values. An interval is defined to be the real compact interval $X = [\underline{x}, \overline{x}]$ where $\underline{x}$ and $\overline{x}$ indicate the lower and upper endpoints. Interval vectors and matrices will will be indicated by bold face lower case and uppercase characters, respectively. It is very difficult to devise an unambiguous symbology for interval mathematics and still preserve a notation that most readers will be familiar with. At times it will be necessary to explicitly state in the text when a quantity is an interval and when it is a point value. A *box* is a vector of intervals such as $\mathbf{x} = (X_1, X_2)$. Geometrically a box is a rectangle in two dimensions and a rectangular solid in three dimensions. For example,

$$X_1 = [1.0, 2.0]$$

is an interval while

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} [1.0,2.0] \\ [-3.2,2.1] \end{pmatrix}$$

is a box. If X and Y are intervals, the binary operations for interval arithmetic are

$$\begin{aligned}
X + Y &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}], \\
X - Y &= [\underline{x} - \overline{y}, \overline{x} - \underline{y}], \\
X \cdot Y &= [min(\underline{x}\,\underline{y}, \underline{x}\,\overline{y}, \overline{x}\,\underline{y}, \overline{x}\,\overline{y}), \\
&\qquad max(\underline{x}\,\underline{y}, \underline{x}\,\overline{y}, \overline{x}\,\underline{y}, \overline{x}\,\overline{y})], \\
X/Y &= [\underline{x}, \overline{x}] \cdot [1/\overline{y}, 1/\underline{y}] \text{ if } 0 \notin Y.
\end{aligned}$$

If $0 \in Y$ then the result for division is either two intervals $[-\infty, a]$ and $[b, +\infty]$ or a single interval $[-\infty, +\infty]$. The arithmetic resulting from division by zero is called *extended interval arithmetic*. The rules for this extended arithmetic are

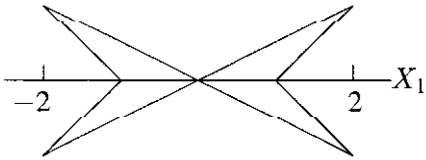

Fig. 1. The solution set of $\mathbf{Ax} = \mathbf{b}$.

not discussed here. Details can be found in Ratschek and Rokne (1988). Interval expressions can be developed for functions of intervals but these are not needed here. Additional interval operations that will be needed later are defined to be

$$| X |= max\{| \underline{x} |, | \overline{x} |\}$$

$$\langle X \rangle = min\{| \underline{x} |, | \overline{x} |\} \text{ if } 0 \notin X, \langle x \rangle = 0 \text{ otherwise.}$$

$$d(X) = \overline{x} - \underline{x}$$

and are called the absolute value, the magnitude, and the diameter of X, respectively.

3. Solutions of Linear Interval Equations

Interval equations have three kinds of solutions. One of these is called the solution set, the second is called an interval solution, and the third is called the hull solution. The solution set of $\mathbf{Ax} = \mathbf{b}$ is defined to be (Ratchek, 1988)

$$\phi = \{x \in R^n : Ax = b \text{ for some } A \in \mathbf{A}, b \in \mathbf{b}\}.$$

Here ϕ represents a set of points in a real number space of dimension n. The set ϕ consists of the points found by solving every point equation $Ax = b$ that can be composed from $\mathbf{Ax} = \mathbf{b}$. In short, any point $x \in \phi$ will satisfy the equation $\mathbf{A}x = \mathbf{b}$. For example, consider the interval equations

$$[2, 4]X_1 + [-1, 1]X_2 = [-3, 3]$$

$$[-1, 1]X_1 + [2, 4]X_2 = [0, 0].$$

The solution set of these equations is shown in Figure 1. Usually the solution set will not be a box and will be quite complicated in higher dimensions. Finding the solution set is very difficult for problems of even modest dimensions and usually is not even considered. Further, it is difficult to know how one might practically use it. For these reasons solutions using the rules of interval arithmetic are sought.

An interval solution, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, is a box found using the rules of interval arithmetic in some solution technique such as Gaussian elimination or Cramer's rule. Such a solution is shown in Figure 2. An unfortunate feature of interval

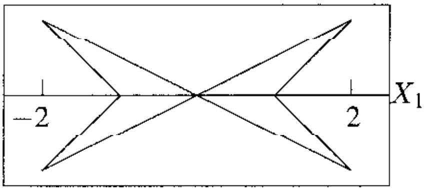

Fig. 2. An interval solution of $\mathbf{Ax} = \mathbf{b}$.

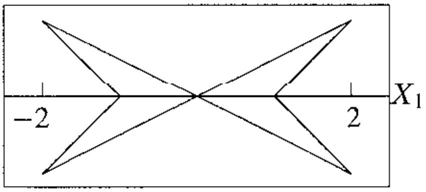

Fig. 3. The hull solution of $\mathbf{Ax} = \mathbf{b}$.

arithmetic is that different solution techniques will yield different boxes. All of the boxes, however, will enclose the solution set if the $\mathbf{A}$ matrix is *regular*, that is if every point matrix A belonging to $\mathbf{A}$ has rank n (Neumaier, 1990).

The *hull* solution, $\mathbf{A}^H\mathbf{b}$, is the smallest box that encloses the solution set. The hull solution for the example system of equations is shown in Figure 3. Finding the hull solution for a set of linear equations is an important problem in interval arithmetic and is usually a difficult task. M-matrices are a special class of matrix for which hull solutions are relatively easy to obtain. The Leontief matrix will most likely be an M-matrix.

4. *M*-matrices

M-matrices were introduced by Ostrowski (1937) and have been widely studied. Economists often call them *Metzler* matrices (Takayama, 1987) for his pioneering work (Metzler, 1945, 1950). The first application to interval matrices was made by Barth and Nuding (1974). If the Leontief matrix is an M-matrix the hull solution is given by

$$\mathbf{L}^{-1}\mathbf{b} = \mathbf{L}^H\mathbf{b} = \begin{cases} [\overline{L}^{-1}\underline{b}, \underline{L}^{-1}\overline{b}] & \text{if } \mathbf{b} \geq 0; \\ [\underline{L}^{-1}\underline{b}, \underline{L}^{-1}\overline{b}] & \text{if } \underline{b} \leq 0 \leq \overline{b}; \\ [\underline{L}^{-1}\underline{b}, \overline{L}^{-1}\overline{b}] & \text{if } \mathbf{b} \leq 0 \end{cases} \tag{1}$$

where $\underline{b}$ and $\overline{b}$ are point vectors formed by using the lower and upper endpoints of the elements of the interval vector $\mathbf{b}$. Likewise $\underline{L}$ and $\overline{L}$ are point matrices formed by using the endpoints of the matrix $\mathbf{L}$. The condition $\mathbf{b} \geq 0$ means that the lower endpoint of every element of $\mathbf{b}$ is nonnegative. Only the condition $\mathbf{b} \geq 0$ need be considered here because final demand must be non-negative. The Leontief matrix will be an M-matrix (Neumaier, 1990) if

(i) $L_{ij} \leq 0$ for all $i \neq j$, and

$$A = \begin{pmatrix} .0987 & .0000 & .0004 & .0036 & .0000 & .0010 & .0006 & .0002 & .0001 \\ .0000 & .0053 & .0003 & .0001 & .0013 & .0000 & .0000 & .0000 & .0001 \\ .0039 & .0013 & .0006 & .0018 & .0079 & .0015 & .0260 & .0046 & .0147 \\ .0089 & .0022 & .0440 & .1018 & .0141 & .0107 & .0018 & .0108 & .0013 \\ .0220 & .0119 & .0355 & .0444 & .0721 & .0315 & .0074 & .0220 & .0247 \\ .0075 & .0022 & .0537 & .0115 & .0040 & .0047 & .0003 & .0067 & .0005 \\ .0569 & .0150 & .0176 & .0129 & .0243 & .0315 & .0451 & .0379 & .0018 \\ .0151 & .0052 & .0790 & .0185 & .0271 & .0234 & .0111 & .0537 & .0017 \\ .0062 & .0026 & .0028 & .0051 & .0048 & .0064 & .0037 & .0078 & .0037 \end{pmatrix}$$

Fig. 4. The technical coefficient matrix for Coconino County.

(ii) $Lu > 0$ for some positive vector u.

While these results are given for point matrices, the application to interval matrices in the next section. For condition (i), note that all off diagonal terms will be of the form $-A_{ij}$ and that the elements of the technical coefficient matrix must be positive. For condition (ii), let $u = (1, 1, \cdots, 1)'$ and form $\mathbf{L}u$. The value of the first element of $\mathbf{L}u$ is

$$1 - A_{11} - A_{12} - \cdots - A_{1n}. \tag{2}$$

This element should be positive because the elements of the technical coefficient matrix represent the proportion of the output form industry 1 used as intermediate goods. Equation (2) can be zero only for a good with no final demand. If so, this good can be aggregated with a closely related good to avoid the problem. If this is not acceptable, a means of working with matrices that are not M-matrices is considered later.

5. An Example

The College of Business Administration at Northern Arizona University has been charged with analyzing the economic impact of the University on Coconino County, Arizona and on the state of Arizona. Because of the short length of time given for the project (4 months) and because sampling costs are prohibitive, we are using a commercially vended product, IMPLAN, to estimate the technical coefficients for both the county and state. IMPLAN adjusts the national coefficients to regional coefficients which provides a source of uncertainty (Olson and Lindall, 1994). IMPLAN produces a technical coefficient matrix for approximately 200 industries depending on the region chosen. These were aggregated into 9 industries for purposes of display. The technical coefficient matrices for Coconino County and the state of Arizona are shown in Figure 4 and Figure 5.

$$A = \begin{pmatrix} .0803 & .0002 & .0046 & .0108 & .0003 & .0013 & .0050 & .0021 & .0008 \\ .0005 & .1995 & .0022 & .0178 & .0032 & .0000 & .0000 & .0000 & .0011 \\ .0068 & .0053 & .0009 & .0021 & .0122 & .0017 & .0226 & .0066 & .0193 \\ .0310 & .0165 & .0482 & .1001 & .0140 & .0193 & .0032 & .0255 & .0018 \\ .0233 & .0466 & .0315 & .0274 & .0722 & .0223 & .0092 & .0235 & .0270 \\ .0255 & .0127 & .0802 & .0231 & .0084 & .0074 & .0006 & .0109 & .0034 \\ .0504 & .0170 & .0192 & .0120 & .0276 & .0286 & .0600 & .0449 & .0029 \\ .0475 & .0350 & .0871 & .0288 & .0406 & .0441 & .0315 & .0844 & .0058 \\ .0070 & .0269 & .0033 & .0061 & .0054 & .0071 & .0043 & .0093 & .0083 \end{pmatrix}$$

Fig. 5. The technical coefficient matrix for Arizona.

The industries are aggregated using IMPLAN Industrial Numbers (IIN). The industry aggregates are agriculture, forestry and fisheries (IIN 1), mining (IIN 29), construction (IIN 48), manufacturing (IIN 58), transportation, communications and utilities (IIN 443), trade (IIN 447), finance, insurance and real estate (IIN 456), services (IIN 463), and government (IIN 512).

Final demand cannot be forecast for either Coconino County or for the state. For that reason various regional multipliers will be calculated for these regions. These are shown in Table I for Coconino County and Table II for Arizona. It is useful to have the Leontief inverse to calculate these multipliers rather than consider solving (essentially) the same problem a number of times. These multipliers assume that the technical coefficient matrix for both Coconino County and the state are subject to uncertainty of $\pm 1\%$. Wage and total income are assumed to have a level of uncertainty of $\pm 0.1\%$.

Neumaier (1990) gives the following results that will be useful for finding the inverse of the interval Leontief matrix

(i) $\mathbf{A}$ will be an M-matrix iff $\underline{A}$ and $\overline{A}$ are M-matrices;
(ii) if $\mathbf{A}$ is an M-matrix then $\mathbf{A}^{-1} = [\overline{A}^{-1}, \underline{A}^{-1}]$;
(iii) if $\mathbf{A}$ is an M-matrix then Gaussian elimination will produce the hull solution of $\mathbf{Ax} = \mathbf{b}$ if $\mathbf{b}$ is nonnegative.

Here $\underline{A}$ and $\overline{A}$ are point matrices formed using the endpoints. The first result indicates the interval Leontief matrix will be an M-matrix for reasonable choices for uncertainty. This should be checked of course. The third provides a means of obtaining the hull inverse. Use an interval version of Gaussian elimination to solve $\mathbf{Lx} = u_i, i = 1, n$, where u_i is a unit vector with the i-th element equal to one and the other elements equal to zero.

The calculations were coded in *C-XSC*, a library of C++ classes (Klatte et al., 1993). This library provides an interval class and interval vector and matrix

TABLE I. Multipliers for Coconino County

Industry IIN	Output Multiplier	Wage Income Multiplier	Total Income Multiplier	Employment Multiplier
1	[1.259,1.266]	[1.319,1.328]	[1.265,1.273]	[1.236,1.242]
29	[1.051,1.053]	[1.150,1.155]	[1.039,1.041]	[1.417,1.428]
48	[1.270,1.277]	[1.472,1.484]	[1.552,1.566]	[1.476,1.488]
58	[1.236,1.243]	[1.304,1.312]	[1.304,1.312]	[1.374,1.384]
433	[1.180,1.185]	[1.213,1.219]	[1.202,1.208]	[1.324,1.333]
447	[1.126,1.130]	[1.079,1.081]	[1.125,1.129]	[1.053,1.055]
456	[1.110,1.114]	[1.222,1.229]	[1.090,1.094]	[1.168,1.173]
463	[1.164,1.168]	[1.118,1.121]	[1.155,1.160]	[1.103,1.106]
512	[1.057,1.059]	[1.017,1.018]	[1.027,1.029]	[1.022,1.024]

TABLE II. Multipliers for Arizona

Industry IIN	Output Multiplier	Wage Income Multiplier	Total Income Multiplier	Employment Multiplier
1	[1.337,1.347]	[1.493,1.507]	[1.375,1.385]	[1.292,1.301]
29	[1.483,1.498]	[2.043,2.073]	[1.794,1.817]	[2.689,2.739]
48	[1.335,1.344]	[1.541,1.555]	[1.613,1.629]	[1.583,1.598]
58	[1.286,1.294]	[1.323,1.333]	[1.329,1.338]	[1.486,1.500]
433	[1.224,1.230]	[1.280,1.288]	[1.230,1.237]	[1.389,1.400]
447	[1.159,1.164]	[1.108,1.111]	[1.140,1.145]	[1.086,1.089]
456	[1.165,1.170]	[1.292,1.301]	[1.170,1.175]	[1.257,1.265]
463	[1.251,1.258]	[1.191,1.197]	[1.221,1.228]	[1.173,1.178]
512	[1.086,1.089]	[1.036,1.038]	[1.050,1.052]	[1.042,1.044]

classes. This interval implementation provides directed rounding. Directed rounding will insure that the results of an interval computation will always enclose the results that would be generated by a computer where no rounding would occur. The values from IMPLAN are input as interval values and all values, including the subjective uncertainty, were treated as interval values.

Rohn (1978, 1980) has derived the conditions under which the input-output system will have economically feasible solutions when the technical coefficients are intervals. He did not directly consider them in context of the M-matrix or consider the Leontief inverse or multipliers. Lorenzen (1985, 1989), Lorenzen and Maas (1989), and Maier (1985) have also considered input-output models with inexact data.

$$A = \begin{pmatrix} 0.16 & 0.26 & 0.03 & 0.05 & 0.13 & 0.13 & 0.19 \\ 0.08 & 0.07 & 0.18 & 0.03 & 0.08 & 0.18 & 0.24 \\ 0.11 & 0.04 & 0.21 & 0.03 & 0.13 & 0.07 & 0.07 \\ 0.17 & 0.02 & 0.05 & 0.21 & 0.16 & 0.09 & 0.06 \\ 0.06 & 0.00 & 0.03 & 0.36 & 0.08 & 0.04 & 0.12 \\ 0.03 & 0.11 & 0.18 & 0.15 & 0.05 & 0.13 & 0.11 \\ 0.25 & 0.32 & 0.18 & 0.13 & 0.18 & 0.20 & 0.01 \end{pmatrix}$$

Fig. 6. Miernyk's Technical Coefficient Matrix

6. What if L is not an M-matrix

Suppose the Leontief matrix is not an M-matrix. Such a case might arise if the household sector is made endogenous. Consider the hypothetical example used in Miernyk's (1965) classical book on input-output analysis, where the technical coefficient matrix (with households endogenous) is shown in Figure 6.

Hansen (1965) has developed a method for dealing with more general matrices. This method, while not guaranteed to produce the hull solution, it is, under conditions mentioned below, guaranteed to enclose the solution set. Hansen's method consists of constructing a preconditioning point matrix L^c with elements

$$L_{ij}^c = (\underline{L}_{ij} + \overline{L}_{ij})/2$$

and solving

$$(L^c)^{-1}\mathbf{L}\mathbf{x} = (L^c)^{-1}\mathbf{b}.$$

The calculation of $(L^c)^{-1}\mathbf{L}$ produces a result close to an identity matrix and should be a *strictly diagonally dominant* matrix. A strictly diagonally dominant matrix is one where

$$\langle A_{ii} \rangle > \sum \mid A_{ij} \mid \quad \text{for} \quad i \neq j.$$

Note that the quantities in the equation above are intervals. Strict diagonal dominance is sufficient for $\mathbf{A}$ to be an H-matrix (Neumaier, 1990), another special case of interval matrices. H-matrices have an important characteristic of M-matrices, they are regular. Recall regularity means that an interval solution will produce a box that will enclose the solution set. Gaussian elimination performed on H-matrices will not produce the hull solution and often leads to severe overestimation. However, experimental evidence shows that Hansen's preconditioning gives H-matrices that yield results close to the hull solution (Wongwises, 1975a, 1975b). At any rate, if the resulting interval values are small then one can be confident of the results.

The interval results for the output multipliers from Miernyk's example are shown in Table III . The levels of uncertainty are arbitrarily assigned to be A plus

TABLE III. Type II Multipliers

Industry	$(1 \pm 0.001)A$ Multiplier	$(1 \pm 0.01)A$ Multiplier	$(1 \pm 0.05)A$ Multiplier
1	[4.812, 4.883]	[4.488,5.221]	[2.867,7.300]
2	[3.794, 3.848]	[3.553,4.098]	[2.354,5.634]
3	[6.340, 6.436]	[5.903,6.892]	[3.715,9.699]
4	[9.200, 9.348]	[8.525,10.052]	[5.125,14.416]
5	[6.035, 6.127]	[5.615,6.565]	[3.508,9.267]
6	[5.711, 5.797]	[5.322,6.202]	[3.377,8.697]

or minus a percentage of A. For those who wish to check these results against Miernyk's should note that his Leontief inverse is slightly in error. The results presented here will enclose Miernyk's results if the correct inverse is used.

7. Inverse Important Coefficients

Some research has suggested that perturbations in certain technical coefficients will produce relatively large changes in the Leontief inverse and in the multipliers. These have been designated as inverse important coefficients (Hewings, 1984). Jensen and West (1980) report that many of the technical coefficients could be replaced with zeros without substantially effecting impact analysis. If resources are available to estimate uncertainty, clearly they should be directed to estimating the inverse important coefficients. Bullard and Sebald (1977) used the Sherman-Morrison Theorem (Sherman, 1950) as an efficient means of determining the importance of individual technical coefficients. Suppose that we have already found an inverse matrix A^{-1}. Now suppose there is a "small" change to some element of A. The Sherman-Morrison formula calculates a new inverse using the elements of the old one. This formula is given by (the quantities here are point quantities) Press et al. (1994) as

$$\begin{aligned}
(A + u \otimes v)^{-1} &= (I + A^{-1} \cdot u \otimes v)^{-1} \cdot A^{-1} \\
&= (I - A^{-1} \cdot u \otimes v + A^{-1} \cdot u \otimes v \cdot A^{-1} \cdot u \otimes v - \ldots) \cdot A^{-1} \\
&= A^{-1} - A^{-1} \cdot u \otimes v \cdot A^{-1}(1 - \lambda + \lambda^2 - \ldots) \\
&= A^{-1} - \frac{(A^{-1} \cdot u) \otimes (v \cdot A^{-1})}{1 + \lambda},
\end{aligned}$$

where

$$\lambda = v \cdot A^{-1} \cdot u$$

and $u \otimes v$ is a matrix whose $i, j-$th element is the product of the $i-$th element of u and the $j-$th element of v. Note the benefit of the Sherman-Morrison formula is that

TABLE IV. Inverse Important Coefficients

Coconino County		Arizona	
Diameter	Element	Diameter	Element
0.0077	(4,4)	0.0189	(2,2)
0.0056	(5,5)	0.0085	(4,4)
0.0049	(8,3)	0.0080	(8,8)
0.0040	(8,8)	0.0060	(8,3)
0.0027	(7,8)	0.0047	(7,7)
0.0021	(8,5)	0.0040	(7,8)
0.0015	(8,6)	0.0035	(8,5)
0.0010	(8,1)	0.0033	(8,6)
0.0008	(8,7)	0.0026	(8,7)
0.0005	(9,8)	0.0019	(9,2)

TABLE V. Inverse Important Coefficients

Coconino County		Arizona	
Diameter	Element	Diameter	Element
0.1984	(4,4)	0.365441	(2,2)
0.1688	(5,5)	0.222852	(8,8)
0.1615	(8,3)	0.188637	(8,3)
0.1390	(8,8)	0.167369	(7,8)
0.1188	(8,5)	0.154153	(8,5)
0.0984	(8,4)	0.141994	(8,6)
0.0943	(8,6)	0.136294	(8,7)
0.0916	(8,7)	0.107747	(9,2)
0.0807	(7,3)	0.100770	(7,2)
0.0790	(9,8)	0.097817	(9,8)

the inverse matrix need only be calculated once. For our purposes the correction terms will be

$$u = c \cdot e_i \text{ and } v = e_j$$

where e_i and e_j are unit vectors and c is some correction factor. This has the effect of adding c to element A_{ij} of A. The results when the correction factor is $A_{ij}(1+0.01)$ are shown in Table IV. The results for a correction factor of $A_{ij}+0.05$ are shown in Table V. These results were obtained by applying the correction factors exhaustively to each term of the matrix, applying the Sherman-Morrison Theorem, and then finding the interval sum of the output multipliers. The diameters of the ten largest sums, $d(\text{sum})$, are presented for both Coconino County and for the state. The element that was perturbed is shown next to the sum.

These tables show some interesting results. For both the relative and absolute changes the largest technical coefficient was associated with the largest sum. Not every large coefficient was associated with a large sum, however. Note that a perturbation of element A_{11} did not produce a large sum for either the county or the state. The results also depend on whether the change is relative or absolute. Finally note the prominence of row eight (services) in all cases. This may be the result of the strong service sector in Arizona.

8. Conclusions

Interval arithmetic offers a way of evaluating the effects of uncertainty in input-output analysis. While these results are not as satisfying as traditional confidence intervals they offer an alternative when the traditional methods cannot be used. With no further information the user will have to arbitrarily assign levels of uncertainty and experiment with the model. Because of the conservative nature of method, if the resulting intervals are small the user can be reasonably confident of the results.

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