Verified numerical computation for elliptic eigenvalue problems and its applications

by
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1 Introduction

Several numerical methods have been proposed to verify the exact eigenvalues for infinite dimensional eigenvalue problems, and in particular the eigenvalues of elliptic operators (*cf.*[2],[15, 16]). In [2], the method is presented to find the upper and lower bounds for eigenvalues by using some test functions, the Rayleigh-Ritz method and the Temple quotient. In [15, 16], given problems are connected with a simple problem, whose explicit eigenvalues are known, by using homotopy method. In this paper, we give a technique that is different from these method. This method is based on the verification method appearing in [25], which is a realization, including uniqueness, of the method studied in [9-14] applicable to nonlinear elliptic boundary value problems.

In Section 2 we apply the numerical method described in [25] to our eigenvalue problems. Using this method we can confirm local uniqueness of *eigenpairs* (*i.e.* pairs of eigenvalues and corresponding eigenfunctions) in a certain set. In the last part of that section, we also confirm the local uniqueness separately of eigenvalues and eigenfunctions as well as the simplicity of the eigenvalues.

In section 3 we describe a method to exclude eigenvalues in order to obtain some informations about index. By this method we can separate the simple eigenvalues and we can also obtain the bound for the Eigenvalue with Smallest Absolute Value (ESAV). This bound plays an important role for rigorous estimates of the norm of the linearized operator of some nonlinear differential equations, which is described in Section 4 in detail.

In Section 5 several numerical examples are presented.

2 *Enclosing* method for eigenvalues with uniqueness property

In this section, we consider a numerical technique to verify the exact eigenvalues and eigenfunctions of second-order elliptic operators in some neighborhood of their approximations. This technique is based on [25] using the Krawczyk-like operator and the error estimates for the C^0 finite element solution. We numerically construct a set containing solutions which satisfies the hypothesis of Banach's fixed point theorem for compact map on a certain Sobolev space.

2.1 Problem and the fixed point formulation

In what follows, let Ω be a bounded convex domain in R^2 and for some integer m, let $H^m(\Omega)$ denote the L^2-Sobolev space of order m on Ω. Then, define $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\}$ with the inner product $< u, v >_{H_0^1} \equiv (\nabla u, \nabla v)_{L^2}$ for $u, v \in H_0^1(\Omega)$, and the norm $\|u\|_{H_0^1} \equiv \|\nabla u\|_{L^2}$ for $u \in H_0^1(\Omega)$, where $(\cdot,\cdot)_{L^2}$ and $\|\cdot\|_{L^2}$ represent the inner product and the norm on $L^2(\Omega)$, respectively.

Now, let S_h be a finite dimensional subspace of $H_0^1(\Omega)$ that depends on h $(0 < h < 1)$. Usually, S_h is taken to be a finite element subspace with mesh size h. Also, let $P_{h0} : H_0^1(\Omega) \longrightarrow S_h$ denote the H_0^1-projection defined by

$$(\nabla(u - P_{h0}u), \nabla v_h)_{L^2} = 0 \qquad \text{for all } v_h \in S_h.$$

We now assume the following approximation property in S_h.

Assumption 1. For any $u \in H^2(\Omega) \bigcap H_0^1(\Omega)$,

$$\inf_{\chi \in S_h} \|u - \chi\|_{H_0^1} \leq C_1 h |u|_{H^2}, \tag{2.1}$$

where

$$|u|_{H^2}^2 \equiv \sum_{i,j=1}^{2} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^2}^2.$$

Here, C_1 is a positive, numerically determined constant which is independent of h.

The following lemma is well known [3].

Lemma 1. *For any ψ in $L^2(\Omega)$, there exists a unique solution $\phi \in H^2(\Omega) \bigcap H_0^1(\Omega)$ of the following Poisson equation:*

$$\begin{cases} -\Delta\phi &= \psi \quad in\ \Omega, \\ \phi &= 0 \quad on\ \partial\Omega. \end{cases} \tag{2.2}$$

Furthermore, there exists a positive constant C_2 satisfying

$$|\phi|_{H^2} \le C_2 \|\psi\|_{L^2}. \tag{2.3}$$

In particular, if Ω is a convex polygonal domain, we can set $C_2 = 1$ ([3]).

We consider the self-adjoint eigenvalue problem

$$\begin{cases} -\Delta u + qu &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $q \in L^\infty(\Omega)$. Since we wish to enclose eigenpairs of this problem, we consider the space $H_0^1(\Omega) \times R$ and define the inner product $< \cdot, \cdot >_{H_0^1 \times R}$ and the norm $\| \cdot \|_{H_0^1 \times R}$ by

$$< w_1, w_2 >_{H_0^1 \times R} \equiv (\nabla u_1, \nabla u_2)_{L^2} + \lambda_1 \lambda_2,$$

$$\|w\|_{H_0^1 \times R} \equiv (\|u\|_{H_0^1}^2 + |\lambda|^2)^{\frac{1}{2}},$$

respectively, where $w_i = (u_i, \lambda_i) \in H_0^1(\Omega) \times R$ $(i = 1, 2)$ and $w = (u, \lambda) \in H_0^1(\Omega) \times R$. Moreover, let I_0 and I be the identity map on $H_0^1(\Omega)$ and $H_0^1(\Omega) \times R$, respectively.

We first normalize the problem (2.4) as

find $(\widehat{u}, \lambda) \in H_0^1(\Omega) \times R$ s.t.

$$\begin{cases} -\Delta \widehat{u} + (q - \lambda)\widehat{u} &= 0, \\ \int_\Omega \widehat{u}^2 \, dx &= 1. \end{cases} \tag{2.5}$$

We then define the projection $P_h : H_0^1(\Omega) \times R \longrightarrow S_h \times R$ by

$$P_h(u, \lambda) \equiv (P_{h0} u, \lambda).$$

Now, let $\widehat{w}_h = (\widehat{u}_h, \widehat{\lambda}_h) \in S_h \times R$ be a finite element solution of (2.5); that is,

$$\begin{cases} (\nabla \widehat{u}_h, \nabla v_h)_{L^2} &= ((\widehat{\lambda}_h - q)\widehat{u}_h, v_h)_{L^2} \quad \text{for all } v_h \in S_h, \\ \displaystyle\int_\Omega \widehat{u}_h^2 \, dx &= 1. \end{cases} \tag{2.6}$$

We used the interval library PROFIL to enclose this solution in very small intervals (*cf.* [4] and Section 5).

We will verify the existence of eigenvalues and eigenfunctions for (2.5) in a neighborhood of $(\bar{u}, \widehat{\lambda}_h)$ satisfying

$$\begin{cases} -\Delta \bar{u} + (q - \widehat{\lambda}_h)\widehat{u}_h &= 0 \quad \text{in } \Omega, \\ \bar{u} &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Note that $\bar{u} \in H^2(\Omega) \bigcap H_0^1(\Omega)$, and $\widehat{w}_h = P_h(\bar{u}, \widehat{\lambda}_h)$. We then have, by (2.5) and (2.7),

$$\begin{cases} -\Delta(\widehat{u} - \bar{u}) &= (\lambda - q)\widehat{u} - (\widehat{\lambda}_h - q)\widehat{u}_h, \\ \displaystyle\int_\Omega \widehat{u}^2 \, dx &= 1. \end{cases} \tag{2.8}$$

Defining $v_1 = \bar{u} - \widehat{u}_h$, we see that $v_1 \in S_h^\perp$, where $S_h^\perp$ represents the orthogonal complement of S_h in $H_0^1(\Omega)$, and we can write

$$\bar{u} = \widehat{u}_h + v_1 \qquad \text{for } \widehat{u}_h \in S_h \text{ and } v_1 \in S_h^\perp.$$

It is known that the *a posteriori* estimates are, in general, much better than the *a priori* estimates, provided that the higher order base functions in S_h are utilized (see [24] for details). Therefore, we use *a posteriori* estimates for v_1 as below.

Let $S_h^* \subset H^1(\Omega)$ be a finite element subspace whose basis consists of the union of the basis on S_h and the base functions having nonzero values on the boundary $\partial\Omega$. Define $\bar{\nabla}\widehat{u}_h \in S_h^* \times S_h^*$, a vector function in two dimension, by the L^2-projection of $\nabla\widehat{u}_h \in L^2(\Omega) \times L^2(\Omega)$ to $S_h^* \times S_h^*$. Then, define $\bar{\Delta}\widehat{u}_h \in L^2(\Omega)$ by $\bar{\Delta}\widehat{u}_h \equiv \nabla \cdot \bar{\nabla}\widehat{u}_h$. We then obtain the following estimation (*cf.*[24]):

$$\|v_1\|_{H_0^1} \le \|\nabla\widehat{u}_h - \bar{\nabla}\widehat{u}_h\| + C_0 h \|\bar{\Delta}\widehat{u}_h + (\widehat{\lambda}_h - q)\widehat{u}_h\|,$$

where $C_0 \equiv C_1 C_2$. Note that in this estimation we used the L^2-estimate of v_1:

$$\|v_1\|_{L^2} \leq C_0 h \|v_1\|_{H_0^1},$$

which is obtained by using the well-known Aubin-Nitsche trick (e.g. [5]).

Now, in order to verify solutions $(\widehat{u}, \lambda)$ of (2.5) near $(\bar{u}, \widehat{\lambda}_h)$, writing

$$\widehat{u} = \bar{u} + \widetilde{u} \ , \ \lambda = \widehat{\lambda}_h + \widetilde{\lambda},$$

we can rewrite (2.8) as

$$\begin{cases} -\Delta \widetilde{u} &= (\widehat{\lambda}_h + \widetilde{\lambda} - q)(\widetilde{u} + \widehat{u}_h + v_1) - (\widehat{\lambda}_h - q)\widehat{u}_h, \\ \int_\Omega (\widetilde{u} + \widehat{u}_h + v_1)^2 \, dx &= 1. \end{cases}$$

Thus using the following compact map on $H_0^1(\Omega) \times R$

$$\begin{aligned} F(\widetilde{u}, \widetilde{\lambda}) \ \equiv \ \Big(& (-\Delta)^{-1} \{ (\widehat{\lambda}_h + \widetilde{\lambda} - q)(\widetilde{u} + \widehat{u}_h + v_1) - (\widehat{\lambda}_h - q)\widehat{u}_h \} \ , \\ & \widetilde{\lambda} + \int_\Omega (\widetilde{u} + \widehat{u}_h + v_1)^2 \, dx - 1 \Big), \end{aligned} \tag{2.9}$$

where $(-\Delta)^{-1}$ is the solution operator for the Poisson equation with homogeneous boundary conditions, we have the fixed point equation

$$w = F(w) \tag{2.10}$$

for $w = (\widetilde{u}, \widetilde{\lambda})$.

2.2 Verification conditions

We now make the following assumption.

Assumption 2. Set $\rho \equiv (-v_1, 0)$ and define $F'(\rho)$ as the Fréchet derivative of F at ρ. Assume that the restriction to $S_h \times R$ of the operator $P_h[I - F'(\rho)] : H_0^1(\Omega) \times R \longrightarrow S_h \times R$ has the inverse

$$[I - F'(\rho)]_h^{-1} : S_h \times R \longrightarrow S_h \times R.$$

The validity of this assumption can be numerically checked in actual computations.

Now, as in [13, 14], we decompose (2.10) into finite and infinite dimensional parts:

$$\begin{cases} P_h w &= P_h F(w), \\ (I - P_h) w &= (I - P_h) F(w). \end{cases} \tag{2.11}$$

We use a Newton-like method only for the finite dimensional part, represented by the first equation in (2.11). First, we define the Newton-like operator

$$N_h(w) \equiv P_h w - [I - F'(\rho)]_h^{-1}(P_h w - P_h F(w)).$$

We next define the operator $T : H_0^1(\Omega) \times R \longrightarrow H_0^1(\Omega) \times R$ as

$$T(w) \equiv N_h(w) + (I - P_h)F(w). \tag{2.12}$$

Then T becomes a compact map on $H_0^1(\Omega) \times R$, and the relation

$$w = T(w) \Longleftrightarrow w = F(w) \tag{2.13}$$

holds.

Now, an arbitrary element $w \in H_0^1(\Omega) \times R$ can be uniquely written as

$$w = (v_h, \mu) + (v_\perp, 0),\ (v_h, \mu) \in S_h \times R,\ (v_\perp, 0) \in S_h^\perp \times \{0\}, \tag{2.14}$$

with

$$v_h = \sum_{j=1}^{M} v_j \phi_j,$$

where $M = \dim\ S_h$, $\{\phi_j\}_{j=1}^M$ is a basis of S_h, $(v_j)_{j=1}^M$ a real vector. For w in (2.14) we use the following notation:

$$\begin{aligned} (w)_i &\equiv |v_i|, i = 1, ..., M, \\ (w)_{M+1} &\equiv \|v_\perp\|_{H_0^1}, \\ (w)_{M+2} &\equiv |\mu|. \end{aligned}$$

We intend to find a fixed point to (2.10) in a set W, referred to as a 'candidate set'. Given a vector $(W_1, ..., W_{M+2})^t$ such that $W_i > 0$ $(i = 1, ..., M+2)$, its candidate set W is defined by

$$W \equiv \{w \in H_0^1(\Omega) \times R \mid (w)_i \leq W_i \ (i = 1, ..., M+2)\}. \tag{2.15}$$

Now let T' be the Fréchet derivative of T. By the method described below we choose two vectors $(Y_1, ..., Y_{M+2})^t, Y_i > 0$ $(i = 1, ..., M+2)$ and $(Z_1, ..., Z_{M+2})^t, Z_i > 0$ $(i = 1, ..., M+2)$ such that

$$(T(0))_i \leq Y_i, \ i = 1, ..., M+2, \tag{2.16}$$

$$(T'(w_1)w_2)_i \leq Z_i, \ i = 1, ..., M+2, \quad \text{for any } w_1, w_2 \in W. \tag{2.17}$$

The verification condition is described in the following theorem.

Theorem 1. *If a candidate set W defined by (2.15) satisfies*

$$Y_i + Z_i < W_i \quad (i = 1, ..., M+2), \tag{2.18}$$

then there exists a fixed point of T in

$$K \equiv \{v \in H_0^1(\Omega) \times R \mid (v)_i \leq Y_i + Z_i \ (i = 1, ..., M+2)\}. \tag{2.19}$$

Moreover, this fixed point is unique within the set W.

In order to prove this theorem, we derive two preliminary lemmas.

Defining the norm $\|\cdot\|_W$ by

$$\|x\|_W \equiv \max_{1 \leq i \leq M+2} \frac{(x)_i}{W_i}, \quad x \in H_0^1(\Omega) \times R, \tag{2.20}$$

we have the following lemma.

Lemma 2. *For each $x \in H_0^1(\Omega) \times R$,*

$$\sup_{w \in W} \|T'(w)x\|_W \leq \max_{1 \leq i \leq M+2} \frac{Z_i}{W_i} \|x\|_W.$$

Proof.

Since W includes a ball centered at 0 and $T'(w)$ is linear for $x \in H_0^1(\Omega) \times R$,

$$\sup_{w \in W} \|T'(w)x\|_W = \|x\|_W \sup_{w \in W} \left\| T'(w) \frac{x}{\|x\|_W} \right\|_W$$

holds. Then, by the definition of $\|\cdot\|_W$, we see that $\frac{x}{\|x\|_W} \in W$ and this implies

$$\sup_{w \in W} \left\| T'(w) \frac{x}{\|x\|_W} \right\|_W \leq \max_{1 \leq i \leq M+2} \frac{Z_i}{W_i}.$$

This proves the lemma. ∎

Lemma 3. *For any* $w_1, w_2 \in W$,

$$\|T(w_1) - T(w_2)\|_W \leq \sup_{s \in [0,1]} \|T'(sw_1 + (1-s)w_2)(w_1 - w_2)\|_W. \qquad (2.21)$$

Proof.

Defining

$$\check{T}(s) \equiv T(sw_1 + (1-s)w_2),$$

apply the mean value theorem to obtain the desired conclusion. ∎

With these two lemmas, we can now prove Theorem 1. As usual, we define the image $J(V)$ of an arbitrary operator J and arbitrary set V as

$$J(V) \equiv \{J(v) | v \in V\}.$$

Proof of Theorem 1.

We first prove that $T(W) \subset W$. By (2.17) and Lemma 3,

$$(T(w) - T(0))_i \leq \sup_{s \in [0,1]} (T'(sw)w)_i \leq Z_i \qquad \text{for all } w \in W$$

holds. Hence, we have

$$\begin{aligned} (T(w))_i &\leq (T(0))_i + (T(w) - T(0))_i \\ &\leq Y_i + Z_i \\ &< W_i, \end{aligned}$$

from which we obtain

$$T(w) \in W.$$

This implies

$$T(W) \subset W.$$

We next prove that, for some $0 < k < 1$,

$$\|T(w_2) - T(w_1)\|_W \le k\|w_2 - w_1\|_W, \qquad \text{for all } w_1, w_2 \in W.$$

Since W is convex, by Lemmas 2 and 3 we have

$$\begin{aligned}
\|T(w_2) - T(w_1)\|_W &\le \sup_{s\in[0,1]} \|T'(sw_2 + (1-s)w_1)(w_2 - w_1)\|_W \\
&\le \sup_{w_3\in W} \|T'(w_3)(w_2 - w_1)\|_W \\
&\le \max_{1\le i\le M+2} \frac{Z_i}{W_i}\|w_2 - w_1\|_W.
\end{aligned}$$

Thus, $Y_i > 0$ $(i = 1, .., M+2)$ and (2.18) imply

$$\frac{Z_i}{W_i} < \frac{Z_i + Y_i}{W_i} \le \ {}^{\exists}k < 1 \ (i = 1, ..., M+2).$$

Therefore, applying Banach's fixed point theorem to T, the theorem is proved. ∎

2.3 Algorithm in a computer

In what follows, we describe the procedure to choose vectors $(Y_1, ..., Y_{M+2})^t$, $Y_i > 0$ $(i = 1, ..., M+2)$, and $(Z_1, ..., Z_{M+2})^t$, $Z_i > 0$ $(i = 1, ..., M+2)$, satisfying (2.16) and (2.17), respectively.

As usual, we define the absolute value of any interval A as

$$|A| \equiv \max_{a\in A} \ |a|.$$

Since

$$\begin{aligned}
T(0) &= N_h(0) + (I - P_h)F(0) \\
&= -[I - F'(\rho)]_h^{-1}(-P_hF(0)) + (I - P_h)F(0) \\
&= [I - F'(\rho)]_h^{-1}P_hF(0) + (I - P_h)F(0)
\end{aligned}$$

holds, for $Y_1, ..., Y_M$ and Y_{M+2} we first determine the interval vector $(\widetilde{Y}_1, ..., \widetilde{Y}_M, \widetilde{Y}_{M+2})^t$ satisfying

$$P_h T(0) = [I - F'(\rho)]_h^{-1} P_h F(0) \subset \left(\sum_{j=1}^{M} \widetilde{Y}_j \phi_j, \widetilde{Y}_{M+2} \right). \tag{2.22}$$

It is then sufficient to set

$$Y_i = |\widetilde{Y}_i| \qquad (i = 1, ..., M, M+2). \tag{2.23}$$

To determine the interval vector $(\widetilde{Y}_1, ..., \widetilde{Y}_M, \widetilde{Y}_{M+2})^t$ satisfying (2.22), we consider the set $\mathcal{Y} \subset S_h \times R$ such that

$$\begin{aligned} \mathcal{Y} \ \equiv \ & \Big\{ y \in S_h \times R \mid \text{for all } i = 1, ..., M+1, \\ & \qquad < [I - F'(\rho)]_h y, \Phi_i >_{H_0^1 \times R} = < P_h F(0), \Phi_i >_{H_0^1 \times R} \Big\}, \end{aligned} \tag{2.24}$$

where $[I - F'(\rho)]_h$ represents the restriction to $S_h \times R$ of the operator $P_h[I - F'(\rho)]$ and we have used the basis $\Phi_1, ..., \Phi_{M+1}$ of $S_h \times R$ given by $\Phi_i \equiv (\phi_i, 0)$ $(i = 1, ..., M)$, $\Phi_{M+1} \equiv (0, 1)$. Clearly, $\mathcal{Y}$ coincides with $P_h T(0)$. In the actual computation, as shown below, we can obtain the interval hull of $\mathcal{Y}$ (denoted by $\boxed{\mathcal{Y}}$) by solving the linear system of equations in (2.24). Then $(\widetilde{Y}_1, ..., \widetilde{Y}_M, \widetilde{Y}_{M+2})^t$ can be determined as

$$\left(\sum_{j=1}^{M} \widetilde{Y}_j \phi_j \ , \widetilde{Y}_{M+2} \right) \equiv \boxed{\mathcal{Y}}. \tag{2.25}$$

Observe that for Φ_i $(1 \le i \le M)$ and $y \equiv (\sum_{j=1}^{M} y_j \phi_j \ , \ y_{M+2})$, we have

$$\begin{aligned} & < [I - F'(\rho)]_h y, \Phi_i >_{H_0^1 \times R} \\ & = \sum_{j=1}^{M} y_j \int_\Omega (\nabla \phi_j \cdot \nabla \phi_i + q \phi_j \phi_i - \widehat{\lambda}_h \phi_j \phi_i) \, dx - y_{M+2} \int_\Omega \widehat{u}_h \phi_i \, dx, \end{aligned} \tag{2.26}$$

and for Φ_{M+1},

$$< [I - F'(\rho)]_h y, \Phi_{M+1} >_{H_0^1 \times R} = -2 \int_\Omega \widehat{u}_h \sum_{j=1}^{M} y_j \phi_j \, dx. \tag{2.27}$$

Moreover, for Φ_i, $1 \le i \le M$,

$$< P_hF(0), \Phi_i >_{H_0^1\times R} = \int_\Omega (\widehat{\lambda}_h - q)v_1\phi_i\, dx \ , \tag{2.28}$$

and for Φ_{M+1},

$$< P_hF(0), \Phi_{M+1} >_{H_0^1\times R} = \int_\Omega (v_1^2 + 2\widehat{u}_h v_1)\, dx. \tag{2.29}$$

Now, in order to obtain the set $\boxed{\mathcal{Y}}$ we define the $(M+1)\times(M+1)$ matrix $G \equiv (g_{ij})_{1\le i,j\le M+1}$ by

$$\begin{aligned} g_{ij} &= (\nabla\phi_i \ , \ \nabla\phi_j) + (\phi_i \ , \ q\phi_j) - \widehat{\lambda}_h(\phi_i \ , \ \phi_j) \ (1 \le i,j \le M), \\ g_{i,M+1} &= -(\widehat{u}_h \ , \ \phi_i) \quad (1 \le i \le M), \\ g_{M+1,j} &= -2(\widehat{u}_h \ , \ \phi_j) \quad (1 \le j \le M), \\ g_{M+1,M+1} &= 0, \end{aligned} \tag{2.30}$$

and the interval vector $\mathbf{r} \equiv ([-r_i, r_i])_{i=1}^{M+1}$ by

$$r_i \equiv \left| \int_\Omega (\widehat{\lambda}_h - q)v_1\phi_i\, dx \right| \quad (i = 1, ..., M),$$

$$r_{M+1} \equiv \left| \int_\Omega (v_1^2 + 2\widehat{u}_h v_1)\, dx \right|.$$

Here, G is invertible by Assumption 2. Then, the interval vector $(\widetilde{Y}_1, ..., \widetilde{Y}_M, \widetilde{Y}_{M+2})^t$ in (2.25) are determined by

$$(\widetilde{Y}_1, ..., \widetilde{Y}_M, \widetilde{Y}_{M+2})^t = G^{-1}\mathbf{r}. \tag{2.31}$$

We can estimate Y_{M+1} by using the following inequality

$$\begin{aligned} \|(I - P_h)T(0)\|_{H_0^1\times R} &= \|(I - P_h)F(0)\|_{H_0^1\times R} \\ &= \|(I_0 - P_{h0})\{(-\Delta)^{-1}(\widehat{\lambda}_h - q)v_1\}\|_{H_0^1} \\ &\le C_0 h\|(\widehat{\lambda}_h - q)v_1\|_{L^2}, \end{aligned}$$

which is derived from Assumption 1 and Lemma 1; that is we can set

$$Y_{M+1} = C_0 h\|(\widehat{\lambda}_h - q)v_1\|_{L^2}. \tag{2.32}$$

Next, we choose a vector $(Z_1, ..., Z_{M+2})^t$ satisfying (2.17).

Since

$$\begin{aligned} T'(w_1)w_2 &= N_h'(w_1)w_2 + (I - P_h)F'(w_1)w_2 \\ &= [I - F'(\rho)]_h^{-1} P_h(F'(w_1)w_2 - F'(\rho)P_h w_2) + (I - P_h)F'(w_1)w_2 \end{aligned}$$

holds, for $Z_1, ..., Z_M$ and Z_{M+2} we first determine the interval vector $(\widetilde{Z}_1, ..., \widetilde{Z}_M, \widetilde{Z}_{M+2})^t$ for all $w_1, w_2 \in W$ satisfying

$$\begin{aligned} P_h T'(w_1)w_2 &= [I - F'(\rho)]_h^{-1} P_h(F'(w_1)w_2 - F'(\rho)P_h w_2) \\ &\subset \left(\sum_{j=1}^{M} \widetilde{Z}_j \phi_j, \widetilde{Z}_{M+2} \right), \end{aligned} \tag{2.33}$$

and then set

$$Z_i = |\widetilde{Z}_i| \qquad (i = 1, ..., M, M+2). \tag{2.34}$$

To determine the interval vector $(\widetilde{Z}_1, ..., \widetilde{Z}_M, \widetilde{Z}_{M+2})^t$ satisfying (2.33), we consider the set $\mathcal{Z} \subset S_h \times R$ such that

$$\begin{aligned} \mathcal{Z} \equiv \Big\{ z \in S_h \times R \mid & \text{ there exist } w_1, w_2 \in W \text{ such that, for all } i = 1, ..., M+1, \\ & < [I - F'(\rho)]_h z, \Phi_i >_{H_0^1 \times R} \\ & = < P_h(F'(w_1)w_2 - F'(\rho)P_h w_2), \Phi_i >_{H_0^1 \times R} \Big\}. \end{aligned} \tag{2.35}$$

In analogy to our treatment of $\mathcal{Y}$, we can obtain the interval hull of $\mathcal{Z}$ (denoted by $\boxed{\mathcal{Z}}$) by solving the linear system of equations in (2.35) using the interval right-hand side, as we now do.

Observe that for Φ_i $(1 \le i \le M)$ and for all $w_1, w_2 \in W$, we have

$$\begin{aligned} & < P_h(F'(w_1)w_2 - F'(\rho)P_h w_2), \Phi_i >_{H_0^1 \times R} \\ & \quad = \int_\Omega (\widehat{\lambda}_h - q)\{(I_0 - P_{h0})u_2\}\phi_i \, dx + \int_\Omega (\lambda_1 u_2 + \lambda_2(u_1 + v_1))\phi_i \, dx, \end{aligned} \tag{2.36}$$

and for Φ_{M+1},

$$< P_h(F'(w_1)w_2 - F'(\rho)P_h w_2), \Phi_{M+1} >_{H_0^1 \times R}$$

$$= 2\int_\Omega (u_1 + v_1)u_2\, dx + 2\int_\Omega \widehat{u}_h(I_0 - P_{h0})u_2\, dx, \tag{2.37}$$

where we have written $w_i = (u_i, \lambda_i)$, $u_i \in H_0^1(\Omega)$, $\lambda_i \in R$ $(i = 1, 2)$.

Therefore, in order to obtain the set $\boxed{\mathcal{Z}}$, we use the matrix G determined by (2.30) and the interval vector $\mathbf{r} \equiv ([-r_i, r_i])_{i=1}^{M+1}$ for which

$$\begin{aligned} r_i \quad &\equiv \sup_{(u_j,\lambda_j)\in W (j=1,2)} \left| \int_\Omega (\widehat{\lambda}_h - q)\{(I_0 - P_{h0})u_2\}\phi_i\, dx \right. \\ &\qquad\qquad \left. + \int_\Omega (\lambda_1 u_2 + \lambda_2(u_1 + v_1))\phi_i\, dx \right|, \end{aligned}$$

$$r_{M+1} \equiv \sup_{(u_j,\lambda_j)\in W (j=1,2)} \left| 2\int_\Omega (u_1 + v_1)u_2\, dx + 2\int_\Omega \widehat{u}_h(I_0 - P_{h0})u_2\, dx \right|.$$

Then we set

$$(\widetilde{Z}_1, ..., \widetilde{Z}_M, \widetilde{Z}_{M+2})^t = G^{-1}\mathbf{r}. \tag{2.38}$$

We can also estimate Z_{M+1} by using the inequality

$$\begin{aligned} \|(I - P_h)T'(w_1)w_2\|_{H_0^1\times R} \quad &= \quad \|(I - P_h)F'(w_1)w_2\|_{H_0^1\times R} \\ &\le \quad C_0 h\|(\widehat{\lambda}_h + \lambda_1 - q)u_2 + \lambda_2(u_1 + v_1 + \widehat{u}_h)\|_{L^2}; \end{aligned}$$

that is, we can set

$$Z_{M+1} = \sup_{(u_i,\lambda_i)\in W\ (i=1,2)} C_0 h\|(\widehat{\lambda}_h + \lambda_1 - q)u_2 + \lambda_2(u_1 + v_1 + \widehat{u}_h)\|_{L^2}. \tag{2.39}$$

Now, we describe an algorithm for finding a vector $(W_1, .., W_{M+1}, W_{M+2})^t$ which satisfies the verification condition (2.18). Since $(Z_i)_{i=1}^{M+2}$ depends on W, we write Z_i as $Z_i(W)$. We use the following iteration method.

Algorithm.
1. Fix a maximum iteration number.
2. Find a vector $(Y_1, ..., Y_{M+2})^t$ satisfying (2.16).
3. Set $W_i \leftarrow Y_i$ $(i = 1, ..., M + 2)$.
4. Find a vector $(Z_1(W), ..., Z_{M+2}(W))^t$ satisfying (2.17).

5. Check the verification condition (2.18);
$Y_i + Z_i(W) < W_i \ \ (i = 1, ..., M+2)$.
If the condition is satisfied, then the verification has succeeded.
If not, set

$$W_i \leftarrow (1+\delta)(Y_i + Z_i) \ \ (i = 1, ..., M+2), \tag{2.40}$$

where δ $(0 < \delta \ll 1)$ represents an inflation parameter (*cf.* [19],[23] etc.), increase the iteration number by 1, and return to step 4.
6. If the maximum iteration number is exceeded without (2.18) being satisfied, the verification has failed.

Now assume that a set W satisfying the hypothesis in Theorem 1 exists. We define

$$U_i \equiv W_i \ (i = 1, ..., M+1), \ \ \Lambda_0 \equiv W_{M+2}$$

and set

$$U \equiv \{u \in H_0^1(\Omega) \mid (u)_i \leq U_i \ (i = 1, ..., M+1)\}, \tag{2.41}$$
$$\Lambda \equiv \{\lambda \in R \mid |\lambda| \leq \Lambda_0\}, \tag{2.42}$$

where

$$\begin{aligned} (u)_i &\equiv |u_i|, \qquad i = 1, ..., M, \\ (u)_{M+1} &\equiv \|u_\perp\|_{H_0^1}, \end{aligned}$$

with

$$u = \sum_{j=1}^{M} u_j \phi_j + u_\perp, \ \sum_{j=1}^{M} u_j \phi_j \in S_h, \ \ u_\perp \in S_h^\perp.$$

Then we have $W = U \times \Lambda$.

By Theorem 1 we are able to confirm the local uniqueness of an *eigenpair* in $U \times \Lambda$. But this does not imply directly that the eigenvalue is unique in Λ, because there may exist another eigenvalue in Λ corresponding to an eigenfunction in a set U' which is different from U.

We therefore must show the local uniqueness individually for each eigenvalue and eigenfunction in Λ and U, respectively.

2.4 Uniqueness of the enclosed eigenvalue

Let U, Λ and W be the sets defined at the end of the previous subsection. Our aim in this subsection is to prove the uniqueness of an eigenvalue in Λ and of an eigenfunction in U separately. We denote the operator $T : H_0^1(\Omega) \times R \longrightarrow H_0^1(\Omega) \times R$ defined by (2.12) as

$$T(u, \lambda) = (T_1(u, \lambda),\ T_2(u, \lambda)), \tag{2.43}$$

where T_1 and T_2 are operators such that

$$T_1 : H_0^1(\Omega) \times R \longrightarrow H_0^1(\Omega), \tag{2.44}$$

$$T_2 : H_0^1(\Omega) \times R \longrightarrow R. \tag{2.45}$$

For a fixed $\lambda \in \Lambda$, define

$$p_\lambda(u) \equiv T_1(u, \lambda). \tag{2.46}$$

Because of the compactness of T, p_λ is also a compact map on $H_0^1(\Omega)$. If (2.18) holds, then we have

$$p_\lambda(u) \in \mathrm{int}(U) \qquad \text{for all } u \in U. \tag{2.47}$$

Now, for $v \in H_0^1(\Omega)$, we write

$$v = \sum_{j=1}^{M} v_j \phi_j + v_\perp,$$

where $\sum_{j=1}^{M} v_j \phi_j \in S_h,\ v_\perp \in S_h^\perp$, and define the norm $\|\cdot\|_U$ by

$$\|v\|_U \equiv \max\left\{ \max_{j=1,..,M} \frac{|v_j|}{U_j}, \frac{\|v_\perp\|_{H_0^1}}{U_{M+1}} \right\}. \tag{2.48}$$

We then have the following lemma.

Lemma 4. *There exists a fixed point of p_λ in U for each $\lambda \in \Lambda$, and this fixed point is unique in U. Moreover, when we denote this fixed point as u_λ, the equality*

$$\int_\Omega (u_\lambda + \bar{u})^2\,dx = 1 \tag{2.49}$$

holds.

Proof.

In the proof of Theorem 1, we proved that, for some $0 < k < 1$,

$$\|T(w_2) - T(w_1)\|_W \le k\|w_2 - w_1\|_W, \qquad \text{for all } w_1, w_2 \in W.$$

Hence, for any $w_1 = (u_1, \lambda), w_2 = (u_2, \lambda)$ in W, we have

$$\|T(u_2, \lambda) - T(u_1, \lambda)\|_W \le k\|(u_2, \lambda) - (u_1, \lambda)\|_W.$$

By definition, it follows that

$$\begin{aligned}
&\|T(u_2, \lambda) - T(u_1, \lambda)\|_W \\
&= \|(p_\lambda(u_2), T_2(u_2, \lambda)) - (p_\lambda(u_1), T_2(u_1, \lambda))\|_W \\
&= \max\left\{\|p_\lambda(u_2) - p_\lambda(u_1)\|_U, \frac{|T_2(u_2, \lambda) - T_2(u_1, \lambda)|}{\Lambda_0}\right\},
\end{aligned}$$

and that

$$\|(u_2, \lambda) - (u_1, \lambda)\|_W = \|(u_2 - u_1, 0)\|_W = \|u_2 - u_1\|_U.$$

Hence, for all $u_1, u_2 \in U$ and for the above k, we have

$$\|p_\lambda(u_2) - p_\lambda(u_1)\|_U \le k\|u_2 - u_1\|_U. \tag{2.50}$$

By (2.47) and (2.50), we can use Banach's fixed point theorem for p_λ. Thus the first part of the lemma is proved.

Denoting the above fixed point as u_λ, we next prove

$$\int_\Omega (u_\lambda + \bar{u})^2\,dx = 1.$$

Noting the action of the finite dimensional part of the operator T, we see that

$$\begin{aligned}(P_{h0}T_1(u,\lambda), T_2(u,\lambda)) &= (P_{h0}u,\lambda) \\ &\quad -[I-F'(\rho)]_h^{-1}\{P_h(u,\lambda)-P_hF(u,\lambda)\}\end{aligned}$$

for $(u,\lambda) \in H_0^1(\Omega)\times R$. Then, making use of the relation $p_\lambda(u) = T_1(u,\lambda)$ we can rewrite the above equality as

$$P_h(u,\lambda) - P_hF(u,\lambda) = [I-F'(\rho)]_h(P_{h0}(u-p_\lambda(u)), \lambda - T_2(u,\lambda)).$$

Comparing the second components on each side of this equality, we have

$$-1 + \int_\Omega (u+\bar{u})^2\,dx = 2\int_\Omega \widehat{u}_h P_{h0}(u-p_\lambda(u))\,dx.$$

In the case $u = u_\lambda$, we have $u_\lambda - p_\lambda(u_\lambda) = 0$, which proves the second part of the lemma. ∎

Now, we obtain the following lemma, which is needed in the proof of the local uniqueness of eigenvalues.

Lemma 5. *Assume that (2.18) in Theorem 1 holds and let $(u^*-\bar{u}, \lambda^*-\widehat{\lambda}_h)$ be a fixed point of T (i.e. (u^*,λ^*) be an eigenpair for (2.5)). If $\lambda^*-\widehat{\lambda}_h \in \Lambda$, then either $u^*-\bar{u}\in U$ or $-u^*-\bar{u}\in U$ holds.*

Proof.

Since there exists a fixed point of $p_{\lambda^*-\widehat{\lambda}_h}$ in U and this fixed point is unique in it by Lemma 4, we write this fixed point as v and define v^* by $v^* = v+\bar{u}$. In what follows we assume that $u^* \neq \pm v^*$.

Since

$$\int_\Omega (v^*)^2\,dx = \int_\Omega (v+\bar{u})^2\,dx = 1$$

holds by Lemma 4, defining

$$\kappa \equiv \int_\Omega u^*v^*\,dx,$$

we have

$$|\kappa| = \left|\int_\Omega u^*v^*\,dx\right| \le \|u^*\|_{L^2}\|v^*\|_{L^2} = 1$$

by Schwarz' inequality. Equality here holds only in the case $u^* = \pm v^*$. Hence our assumption $u^* \neq \pm v^*$ implies $|\kappa| \neq 1$.

Now, for each $t \in R$ we define

$$g(t) \equiv \xi(t)u^* + \eta(t)v^*, \tag{2.51}$$

where the functions $\xi(t)$ and $\eta(t)$ are defined by

$$\xi(t) \equiv \frac{1}{\sqrt{2}}\left(\frac{\cos t}{\sqrt{1+\kappa}} + \frac{\sin t}{\sqrt{1-\kappa}}\right), \eta(t) \equiv \frac{1}{\sqrt{2}}\left(\frac{\cos t}{\sqrt{1+\kappa}} - \frac{\sin t}{\sqrt{1-\kappa}}\right).$$

Then we obtain

$$\|g(t)\|_{L^2} = 1$$

by a straightforward calculation. Moreover, we can prove that $g(t) - \bar{u}$ is a fixed point of $p_{\lambda^* - \widehat{\lambda}_h}$ for all $t \in R$ through some simple calculations. In particular, we have

$$g(t_1) = v^*, \quad \text{for } t_1 \equiv \sin^{-1}\left(-\frac{\sqrt{1-\kappa}}{\sqrt{2}}\right).$$

Since $g(t)$ is continuous in t and not constant around t_1, and since the fixed point of $p_{\lambda^* - \widehat{\lambda}_h}$ exists in the interior of U by (2.47), there exists a real number $t^* \neq t_1$ sufficiently close to t_1 satisfying

$$g(t^*) - \bar{u} \neq v \text{ and } g(t^*) - \bar{u} \in U.$$

This contradicts the uniqueness of the fixed point of $p_{\lambda^* - \widehat{\lambda}_h}$ in U. Consequently,

$$u^* = v^* \text{ or } u^* = -v^*,$$

which implies

$$u^* - \bar{u} \in U \text{ or } -u^* - \bar{u} \in U.$$

Moreover, if both $u^* - \bar{u} \in U$ and $-u^* - \bar{u} \in U$ hold, then both u^* and $-u^*$ are eigenfunctions corresponding to the eigenvalue λ^* and satisfy $\int_\Omega (u^*)^2\, dx = \int_\Omega (-u^*)^2\, dx = 1$. Therefore, both $(u^* - \bar{u}, \lambda^* - \widehat{\lambda}_h)$ and $(-u^* - \bar{u}, \lambda^* - \widehat{\lambda}_h)$ are fixed points of T in $U \times \Lambda$. Hence Theorem 1 leads us to $u^* = -u^*$, and therefore $u^* = 0$. This is a contradiction. Thus we find that

$$u^* - \bar{u} \in U \Longrightarrow -u^* - \bar{u} \notin U.$$

In a similar manner, we can also show that

$$-u^* - \bar{u} \in U \Longrightarrow u^* - \bar{u} \notin U.$$

Therefore the lemma is proved. ∎

We next prove two additional lemmas, which are needed in the proof of the local uniqueness of eigenfunctions.

For a fixed $u \in U$, define

$$p_u(\lambda) \equiv T_2(u, \lambda), \tag{2.52}$$

where T_2 is the same as in (2.43). Then (2.18) yields

$$p_u(\lambda) \in \mathrm{int}(\Lambda) \qquad \text{for all } \lambda \in \Lambda. \tag{2.53}$$

We have the following lemma.

Lemma 6. *There exists a fixed point of p_u in Λ for each $u \in U$, and this fixed point is unique in Λ.*

Proof.

In the proof of Theorem 1 we proved that, for some $0 < k < 1$,

$$\|T(w_2) - T(w_1)\|_W \leq k\|w_2 - w_1\|_W, \qquad \text{for all } w_1, w_2 \in W.$$

Therefore, for any $w_1 = (u, \lambda_1), w_2 = (u, \lambda_2)$ in W we have

$$\|T(u, \lambda_2) - T(u, \lambda_1)\|_W \leq k\|(u, \lambda_2) - (u, \lambda_1)\|_W.$$

Observe that, by definition, we have

$$\|T(u, \lambda_2) - T(u, \lambda_1)\|_W = \max\left\{\|T_1(u, \lambda_2) - T_1(u, \lambda_1)\|_U, \frac{|p_u(\lambda_2) - p_u(\lambda_1)|}{\Lambda_0}\right\}$$

and

$$\|(u,\lambda_2)-(u,\lambda_1)\|_W = \frac{|\lambda_2-\lambda_1|}{\Lambda_0}.$$

Hence for all $\lambda_1, \lambda_2 \in \Lambda$ and for the above k, we have

$$|p_u(\lambda_2)-p_u(\lambda_1)| \le k|\lambda_2-\lambda_1|. \tag{2.54}$$

Since we can again use Banach's fixed point theorem for p_u, by (2.53) and (2.54) the lemma is proved. ∎

With this result we are able to prove the following lemma.

Lemma 7. *Assume that (2.18) in Theorem 1 holds and let* $(u^*-\bar{u}, \lambda^*-\widehat{\lambda}_h)$ *be a fixed point of* T. *If* $u^*-\bar{u} \in U$, *then we have* $\lambda^*-\widehat{\lambda}_h \in \Lambda$.

Proof.

We denote a fixed point of $p_{u^*-\bar{u}}$ in Λ by μ, which is unique in Λ by Lemma 6, and define μ^* by $\mu^* = \mu + \widehat{\lambda}_h$.

Assume that $\lambda^* - \widehat{\lambda}_h \neq \mu$ holds. Defining $\nu(t)$ for each $t \in R$ as

$$\nu(t) \equiv (1-t)\lambda^* + t\mu^*,$$

through some simple calculations we can find that $\nu(t) - \widehat{\lambda}_h$ is a fixed point of $p_{u^*-\bar{u}}$ for all $t \in R$. In particular, $\nu(1) = \mu^*$. Since the fixed point of $p_{u^*-\bar{u}}$ exists in the interior of Λ by (2.53), by the property of $\nu(t)$ there exists a real number $t^{**} \neq 1$ sufficiently close to 1 such that

$$\nu(t^{**}) - \widehat{\lambda}_h \neq \mu \text{ and } \nu(t^{**}) - \widehat{\lambda}_h \in \Lambda.$$

This contradicts the uniqueness of the fixed point of $p_{u^*-\bar{u}}$ in Λ. Therefore we have

$$\lambda^* - \widehat{\lambda}_h = \mu \in \Lambda,$$

and the lemma is proved. ∎

From Theorem 1, and Lemmas 5 and 7 we can obtain the following theorem which is the main result of this section.

Theorem 2. *If a set* $W = U \times \Lambda$ *satisfies the conditions in Theorem 1, then we have*

i) ${}^{\exists 1}u^*$ *: eigenfunction s.t.* $u^* - \bar{u} \in U$, $\displaystyle\int_\Omega (u^*)^2\,dx = 1$,

ii) ${}^{\exists 1}\lambda^*$ *: eigenvalue s.t.* $\lambda^* - \widehat{\lambda}_h \in \Lambda$,

iii) $F(u^* - \bar{u}, \lambda^* - \widehat{\lambda}_h) = (u^* - \bar{u}, \lambda^* - \widehat{\lambda}_h)$,

iv) λ^* *: geometric simple eigenvalue.*

Proof.

i) The existence of the eigenfunction u^* satisfying $u^* - \bar{u} \in U$ and $\int_\Omega (u^*)^2\,dx = 1$ is confirmed by Theorem 1. We now prove its uniqueness.

Assume that there exists an eigenfunction v^* which is distinct from u^* and satisfies $v^* - \bar{u} \in U$ and $\int_\Omega (v^*)^2\,dx = 1$. Let λ^* and μ^* be the eigenvalues corresponding to the eigenfunctions u^* and v^*, respectively. Then by Lemma 7 we have $\lambda^* - \widehat{\lambda}_h$ and $\mu^* - \widehat{\lambda}_h \in \Lambda$. Therefore Theorem 1 implies

$$\lambda^* = \mu^* \text{ and } \ u^* = v^*,$$

which is a contradiction.

ii) Since we can show the existence of the eigenvalue λ^* satisfying $\lambda^* - \widehat{\lambda}_h \in \Lambda$ by Theorem 1, we need only prove its uniqueness.

Assume that there exists another eigenvalue μ^* which is not equal to λ^* and satisfies $\mu^* - \widehat{\lambda}_h \in \Lambda$. Then the normalized eigenfunction v^* corresponding to μ^* satisfies either $v^* - \bar{u} \in U$ or $-v^* - \bar{u} \in U$ by Lemma 5. Similarly, the normalized eigenfunction corresponding to λ^* also satisfies either $u^* - \bar{u} \in U$ or $-u^* - \bar{u} \in U$. Hence $\mu^* = \lambda^*$ holds by Theorem 1, which is a contradiction.

iii) It is obvious by Theorem 1, and i) and ii) above.

iv) Assuming that λ^* is not geometric simple, there exist two eigenfunctions which correspond to λ^* and are linearly independent. We can normalize these eigenfunctions by

$$\int_\Omega (u^*)^2\,dx = \int_\Omega (v^*)^2\,dx = 1.$$

Note that u^* and v^* are also linearly independent after normalization. We then have both

$$u^* - \bar{u} \in U \text{ or } -u^* - \bar{u} \in U$$

and

$$v^* - \bar{u} \in U \text{ or } -v^* - \bar{u} \in U$$

by Lemma 5. Therefore Theorem 1 leads us to conclude that either $u^* = v^*$ or $u^* = -v^*$. However, this contradicts the linear independent nature of u^* and v^*. ∎

3 *Excluding* method for eigenvalues

3.1 Motivation

In this section, we will mention about a verification method of *excluding* eigenvalues. One of the reason why we need to exclude eigenvalues is that we want to know some indices of eigenvalues. Although we can enclose some eigenvalues by the method described in Section 2, we can obtain nothing about its indices. In order to obtain some informations about indices, we need to check that an interval contains no eigenvalues.

The other reason is an application to the verification of the solutions for the following nonlinear elliptic boundary value problems (*cf.* [12]):

$$\begin{cases} -\Delta u &= f(u) \quad \text{in } \Omega, \\ u &= 0 \quad\quad \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where some appropriate assumptions are given on the nonlinear map f. Setting $F_0 \equiv (-\Delta)^{-1} f$, we can rewrite (3.1) as the fixed point equation:$u = F_0(u)$ on $H_0^1(\Omega)$. Our verification process is based upon the following Newton-like method to $(I_0 - F_0)(u) = 0$:

$$u_n = u_{n-1} - [I_0 - F_0'(u_h)]^{-1}(I_0 - F_0)(u_{n-1}),$$

where $F_0'(u_h)$ is the Fréchet derivative of F_0 at the approximate solution u_h. Up to now, instead of estimating $[I_0 - F_0'(u_h)]^{-1}$ directly, we divided $(I_0 - F_0)(u) = 0$ into finite and infinite dimensional parts, and we used the Newton-like method only in the former part. But if we estimate ESAV of the following eigenvalue problems

$$(-\Delta)[I_0 - F_0'(u_h)](u) = \lambda u,$$

that is,

$$-\Delta u - f'(u_h)u = \lambda u, \tag{3.2}$$

then we can directly estimate $[I_0 - F_0'(u_h)]^{-1}$. Namely, setting $q = -f'(u_h)$ in (2.4), if we verify the ESAV, we can apply the Newton-like method to the infinite dimensional problems(*cf.* [17]). In order to estimate the ESAV rigorously, we use the verified estimation of the bound of it by excluding eiganvalues.

3.2 Verification conditions

Now, for $q \in L^\infty(\Omega)$, we consider the following self-adjoint eigenvalue problem:

$$\begin{cases} -\Delta u + qu &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

In order to estimate the ESAV, we consider whether or not a given interval contains any eigenvalues of (3.3). First, we assume that the ESAV is negative and we begin a procedure to determine a bound of ESAV moving from zero in the negative direction.

We consider a sufficiently narrow interval $\Lambda_i = (a_i, a_{i-1})$, where a_i $(i \geq 1)$ are negative real numbers and $a_0 = 0$ (see Figure 1 (a)). For a fixed i and $\lambda \in \Lambda_i$, we consider (3.3) as the second-order elliptic boundary value problem. Then (3.3) has a trivial solution $u = 0$. Therefore, for any $\lambda \in \Lambda_i$, if we validate the uniqueness of the solution in (3.3) by the method described below, it implies that λ is not an eigenvalue of (3.3); that is, there is no eigenvalue of (3.3) in Λ_i. If we fail to validate the uniqueness in an interval Λ_j , then we set $\lambda^* \equiv \inf\Lambda_{j-1}$ (see Figure 1 (b)). Next, we start this procedure from zero and move the positive direction. In this case, if we fail to validate the uniqueness in an interval Γ_k, then we set $\lambda^{**} \equiv \sup\Gamma_{k-1}$ (see Figure 1 (c)). Note that we can terminate this process when moving in the positive direction after $\inf\Gamma_l > |\lambda^*|$ for some interval Γ_l. By comparing the absolute value of $\lambda^*$and λ^{**}, we can determine a lower bound for the ESAV.

If we fail to validate the uniqueness of the solution $u = 0$ in interval Λ_1 or interval Γ_1, this implies that we could not get a bound of the ESAV as positive values. In this case, we must refine our method, for example, using a smaller mesh size or higher order base functions in S_h, etc. However, there is the possibility that (3.3) really has 0 as an eigenvalue in such a case.

Now, we describe a method to validate the uniqueness of solutions to (3.3) for a fixed $\lambda \in \Lambda_i$. Using the compact map on $H_0^1(\Omega)$

$$F_\lambda(u) \equiv (-\Delta)^{-1}(\lambda - q)u,$$

we can rewrite (3.3) as

$$u = F_\lambda(u). \tag{3.4}$$

Then we set

$$(N_{h0})_\lambda(u) \equiv P_{h0}u - [I_0 - F_\lambda]_{h0}^{-1}(P_{h0}u - P_{h0}F_\lambda(u)),$$

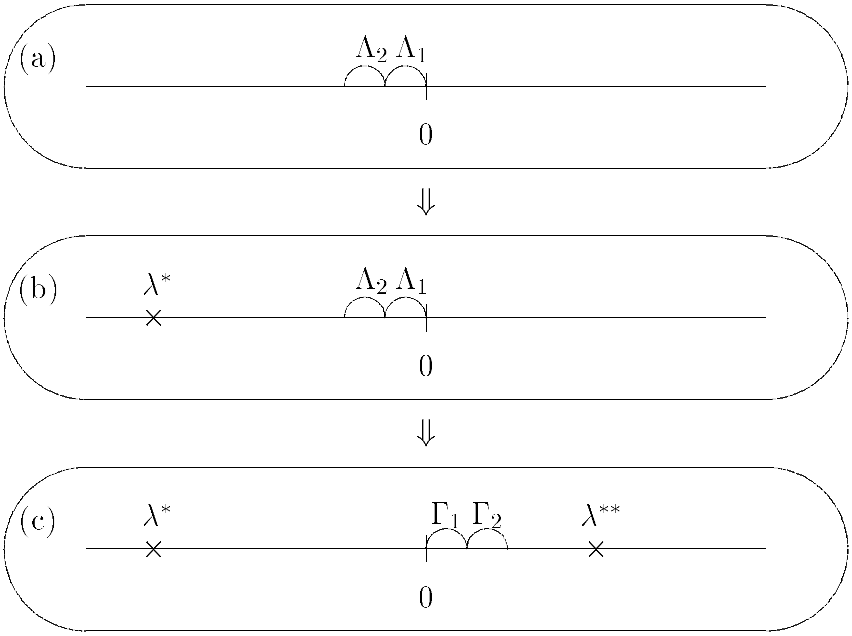

Figure 1: Process of verification

$$T_\lambda(u) \equiv (N_{h0})_\lambda(u) + (I_0 - P_{h0})F_\lambda(u),$$

where we assume that the restriction to S_h of the operator $P_{h0}[I_0 - F_\lambda] : H_0^1(\Omega) \longrightarrow S_h$ has an inverse, $[I_0 - F_\lambda]_{h0}^{-1}$. Then T_λ is a compact linear map on $H_0^1(\Omega)$, and the equivalence relation

$$u = T_\lambda(u) \Longleftrightarrow u = F_\lambda(u) \tag{3.5}$$

holds. Thus we have the following theorem:

Theorem 3. *If there exists a non-empty, closed, bounded and convex set $U \subset H_0^1(\Omega)$ satisfying $T_\lambda(U) \overset{\circ}{\subset} U$, then there exists a unique solution $u \in H_0^1(\Omega)$ of $u = F_\lambda(u)$.*

Here, $V_1 \overset{\circ}{\subset} V_2$ implies $\bar{V_1} \subset \overset{\circ}{V_2}$ for any sets V_1, V_2.

Proof.

Using Schauder's fixed point theorem, there exists u in $\overset{\circ}{U}$ satisfying

$$u = T_\lambda(u), \tag{3.6}$$

and by (3.5) it is equivalent to $u = F_\lambda(u)$.

Since T_λ is a linear operator and $T_\lambda(u) = u$ holds, for any $c \in R$ we have

$$\begin{aligned} T_\lambda(cu) &= cT_\lambda(u) \\ &= cu. \end{aligned} \tag{3.7}$$

If $u \neq 0$, we can choose $\hat{c} \in R$ satisfying

$$\hat{c}u \in \partial U.$$

But this contradicts with $T_\lambda(U) \overset{\circ}{\subset} U$ and (3.7). Therefore $u = 0$. That is, $u = 0$ is a unique solution of $u = F_\lambda(u)$. ∎

By Theorem 3, if there exists a closed, bounded and convex set $U \subset H_0^1(\Omega)$ satisfying $\bigcup_{\lambda \in \Lambda_i} T_\lambda U \overset{\circ}{\subset} U$ then it follows that the interval Λ_i contains no eigenvalues of (3.3). We use interval approaches to verify the condition $\bigcup_{\lambda \in \Lambda_i} T_\lambda U \overset{\circ}{\subset} U$ (*cf.*[14]).

3.3 Algorithm in a computer

We now describe the actual computational procedures used to verify the condition in Theorem 3.

For an interval vector $\mathbf{B}_0 \equiv (B_j^{(0)})_{j=1}^M$ and a strictly positive real number α_0, we write

$$U_h = \sum_{j=1}^{M} B_j^{(0)} \phi_j,$$

$$U_\perp = \{\psi \in S_h^\perp \mid \|\psi\|_{H_0^1} \leq \alpha_0\},$$

where $M = \dim S_h$, $\{\phi_j\}_{j=1}^M$ is a basis of S_h. And we define U as

$$U \equiv U_h \oplus U_\perp.$$

Then the verification conditions are written as follows:

$$\begin{cases} (N_{h0})_\lambda(U) \overset{\circ}{\subset} U_h \\ (I_0 - P_{h0})F_\lambda(U) \overset{\circ}{\subset} U_\perp. \end{cases} \tag{3.8}$$

If an interval vector $\mathbf{B} \equiv (B_j)_{j=1}^{M}$ such that

$$\sum_{j=1}^{M} B_j \phi_j \supset (N_{h0})_\lambda U \tag{3.9}$$

and a real number α defined by

$$\alpha \equiv C_0 h \sup_{v \in U} \|(\lambda - q)v\|_{L^2}$$

satisfy the conditions

$$\begin{cases} \mathbf{B} & \overset{\circ}{\subset} & \mathbf{B_0}, \\ \alpha & < & \alpha_0, \end{cases} \tag{3.10}$$

then the verification conditions (3.8) are satisfied, and there exists a unique solution of (3.3) in U for a fixed $\lambda \in \Lambda_i$. Note that the inclusion in the former part of (3.10) is meant componentwise.

Next we derive a necessary and sufficient condition for (3.10) which is simpler than (3.10). If $q \equiv \lambda$, (3.3) has the only solution $u = 0$. Therefore we assume that $q \not\equiv \lambda$.

If we represent an element v in U as $v = v_h + v_\perp$ for $v_h \in U_h$ and $v_\perp \in U_\perp$, we have

$$\begin{aligned} (N_{h0})_\lambda(v) &= (N_{h0})_\lambda(v_h + v_\perp) \\ &= P_{h0}(v_h + v_\perp) - [I_0 - F_\lambda]_{h0}^{-1}(P_{h0}(v_h + v_\perp) - P_{h0}F_\lambda(v_h + v_\perp)) \\ &= v_h - [I_0 - F_\lambda]_{h0}^{-1}(v_h - P_{h0}F_\lambda(v_h + v_\perp)). \end{aligned}$$

That is,

$$\begin{aligned} [I_0 - F_\lambda]_{h0}(N_{h0})_\lambda(v) &= [I_0 - F_\lambda]_{h0} v_h - (v_h - P_{h0}F_\lambda(v_h + v_\perp)) \\ &= P_{h0}F_\lambda(v_h + v_\perp) - P_{h0}F_\lambda(v_h) \\ &= P_{h0}F_\lambda(v_\perp). \end{aligned} \tag{3.11}$$

To calculate the interval vector $(B_j)_{j=1}^{M}$ satisfying (3.9), we consider a set X such that

$$\begin{aligned} X \equiv \Big\{ & x \in S_h \mid \text{there exist } v_\perp \in U_\perp \text{ such that, for all } i = 1, ..., M, \\ & \quad < [I_0 - F_\lambda]_{h0} x, \phi_i >_{H_0^1} = < P_{h0}F_\lambda(v_\perp), \phi_i >_{H_0^1} \Big\}. \end{aligned} \tag{3.12}$$

In an actual computation, as shown below, we can obtain the interval hull of X (denoted by $\boxed{X}$) by solving the linear system of equations in (3.12) using the interval right-hand side. Therefore we can set

$$\sum_{j=1}^{M} B_j \phi_j \equiv \boxed{X}. \tag{3.13}$$

Observe that for ϕ_i $(1 \le i \le M)$ and $x \equiv \sum_{j=1}^{M} x_j \phi_j$, we have

$$\begin{aligned} &< [I_0 - F_\lambda]_{h0} x, \phi_i >_{H_0^1} \\ &= \sum_{j=1}^{M} x_j \left(\int_\Omega \nabla \phi_j \cdot \nabla \phi_i \, dx + \int_\Omega q \phi_j \phi_i \, dx - \lambda \int_\Omega \phi_j \phi_i \, dx \right), \end{aligned} \tag{3.14}$$

and for all $v_\perp \in U_\perp$,

$$< P_{h0} F_\lambda(v_\perp), \phi_i >_{H_0^1} = \int_\Omega (\lambda - q) v_\perp \phi_i \, dx. \tag{3.15}$$

Therefore, in order to obtain the set $\boxed{X}$, we define the $M \times M$ matrix $G^{(\lambda)} \equiv (g_{ij}^{(\lambda)})_{1 \le i,j \le M}$, which is dependent on λ, by

$$g_{ij}^{(\lambda)} = (\nabla \phi_i \ , \ \nabla \phi_j)_{L^2} + (\phi_i \ , \ q\phi_j)_{L^2} - \lambda(\phi_i \ , \ \phi_j)_{L^2} \quad (1 \le i \ , \ j \le M)$$

and the interval vector $\mathbf{r}$ by

$$\mathbf{r} \equiv ([-r_i, r_i]), \ \ r_i \equiv C_0 h \|(\lambda - q)\phi_i\|_{L^2}.$$

Since we supposed that $[I_0 - F_\lambda]_{h0}^{-1}$ exists, $G^{(\lambda)}$ is invertible. Then, the interval coefficients B_j in (3.9) are determined by

$$\mathbf{B} = \alpha_0 (G^{(\lambda)})^{-1} \mathbf{r}. \tag{3.16}$$

Note that $\mathbf{B}$ is obtained as the interval hull of the set $\{x \in R^M \mid G^{(\lambda)} x = \alpha_0 r, \text{ for all } r \in \mathbf{r}\}$ by the usual interval arithmetic. We can estimate α by using, for example, the triangle inequality

$$\begin{aligned} \alpha &= C_0 h \sup_{v \in U} \|(\lambda - q) v\|_{L^2} \\ &\le C_0 h \left\{ \sup_{v_h \in U_h} \|(\lambda - q) v_h\|_{L^2} + \sup_{v_\perp \in U_\perp} \|(\lambda - q) v_\perp\|_{L^2} \right\} \\ &\le C_0 h \sup_{v_h \in U_h} \|(\lambda - q) v_h\|_{L^2} + C_0^2 h^2 \alpha_0 \|\lambda - q\|_\infty, \end{aligned}$$

where $\| \cdot \|_\infty$ represents the L^∞-norm on Ω. Then the following theorem holds.

Theorem 4. *The conditions (3.10) are equivalent to the following inequality:*

$$C_0h \sup_{z\in(G^{(\lambda)})^{-1}\mathbf{r}} \|(\lambda-q)\mathbf{\Phi}^t\cdot z\|_{L^2} + C_0^2h^2\|\lambda-q\|_\infty < 1, \tag{3.17}$$

where $\mathbf{\Phi} \equiv (\phi_i)_{i=1}^M$.

Proof.

First, we suppose that (3.10) hold. Then, the verification conditions (3.10) are written as

$$\begin{cases} \alpha_0(G^{(\lambda)})^{-1}\mathbf{r} & \overset{\circ}{\subset} & \mathbf{B_0}, \\ C_0h \sup_{B_0\in\mathbf{B_0}} \|(\lambda-q)\mathbf{\Phi}^t\cdot B_0\|_{L^2} + C_0^2h^2\alpha_0\|\lambda-q\|_\infty & < & \alpha_0. \end{cases} \tag{3.18}$$

Thus we have

$$\begin{aligned} \alpha_0 &> C_0h \sup_{B_0\in\mathbf{B_0}} \|(\lambda-q)\mathbf{\Phi}^t\cdot B_0\|_{L^2} + C_0^2h^2\alpha_0\|\lambda-q\|_\infty \\ &\geq C_0h \sup_{z\in(G^{(\lambda)})^{-1}\mathbf{r}} \|(\lambda-q)\mathbf{\Phi}^t\cdot \alpha_0 z\|_{L^2} + C_0^2h^2\alpha_0\|\lambda-q\|_\infty \\ &= \alpha_0\left(C_0h \sup_{z\in(G^{(\lambda)})^{-1}\mathbf{r}} \|(\lambda-q)\mathbf{\Phi}^t\cdot z\|_{L^2} + C_0^2h^2\|\lambda-q\|_\infty\right), \end{aligned}$$

and hence, (3.17) holds.

On the other hand, suppose that (3.17) holds. Then defining

$$\epsilon \equiv 1 - C_0h \sup_{z\in(G^{(\lambda)})^{-1}\mathbf{r}} \|(\lambda-q)\mathbf{\Phi}^t\cdot z\|_{L^2} - C_0^2h^2\|\lambda-q\|_\infty,$$

$\epsilon > 0$ holds by (3.17). Since we assumed that $q \not\equiv \lambda$, we define the interval vector $\mathbf{b}_0$ as

$$\mathbf{b}_0 \equiv \frac{\epsilon}{\tau(M+1)C_0h\|\lambda-q\|_\infty}[\mathbf{1}],$$

where $[\mathbf{1}] \equiv ([-1,1],...,[-1,1])^t$, $\tau \equiv \max_{1\leq i\leq M} \|\phi_i\|_{L^2}$ and write

$$\mathbf{B}_0 \equiv \alpha_0(G^{(\lambda)})^{-1}\mathbf{r} + \alpha_0\mathbf{b}_0. \tag{3.19}$$

Then we have

$$\alpha_0(G^{(\lambda)})^{-1}\mathbf{r} \overset{\circ}{\subset} \mathbf{B_0},$$

and by straightforward calculation, we find that the following inequality holds:

$$C_0 h \sup_{B_0 \in \mathbf{B_0}} \|(\lambda - q)\mathbf{\Phi}^t \cdot B_0\|_{L^2} + C_0^2 h^2 \alpha_0 \|\lambda - q\|_\infty < \alpha_0.$$

This proves the theorem. ∎

Remark 3.1

We can choose α_0 and $\mathbf{B_0}$ arbitrarily if they satisfy the relation (3.19). This arbitrarity comes from the linearity of F_λ, which enables us to calculate $\mathbf{B}$ independently of $\mathbf{B_0}$ (See (3.11)).

By Theorem 4, if each $\lambda \in \Lambda_i$ satisfies (3.17), it follows that there is no eigenvalue of (3.3) in Λ_i. In our previous method [14], it is necessary to check the condition (3.10) in the sense of *intervals*. Actually, in that case we often failed to verify the uniqueness of the trivial solution on an interval in which no eigenvalues were contained. In the case of present method, the condition (3.17) can be easily checked for all $\lambda \in \Lambda_i$. Thus, computational efficiency is greatly increased in our new method. This has been confirmed by actual computations.

4 Applications to nonlinear elliptic problems

There are two known methods to verify the existence of solutions of nonlinear elliptic equations, Nakao's method [9-14] and Plum's method [17, 18]. In Nakao's method, one decomposes the equation into finite and infinite dimensional parts, and the former is processed by finite element approximations, while the latter is treated using constructive error estimates. Then, Newton-like iterations for some function sets are executed to find a solution.

In Plum's method, the norm of the inverse of the linearized operator of the original differential equation is evaluated by rigorously estimating the Eigenvalue with Smallest Absolute Value (ESAV) of this linearized operator, and the solution in question is enclosed near an approximate solution by checking a condition of the Newton-Kantorovich type in an infinite dimensional space. Therefore, in Plum's method the estimation of the ESAV plays an essentially important role.

In this section, we propose an alternative approach to this problem consisting of a mixture of Nakao's and Plum's verification methods. More precisely, we use a form of Plum's method with local uniqueness for the basic formulation (*cf.* [25]), and for the estimation of the ESAV of the linearized operator, we use Nakao's method described in Section 3. By applying such a mixed approach, we can obtain a new verification method which includes all the advantages of the two previous methods and none of their disadvantages.

4.1 Statement of the problem and the fixed point formulation

We consider the nonlinear elliptic boundary value problems of the form

$$\begin{cases} -\Delta u &= f(u) \quad \text{in } \Omega, \\ \quad u &= 0 \quad\quad\ \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where we suppose that f satisfies the following assumptions:

A1. $f : H_0^1(\Omega) \longrightarrow L^2(\Omega)$ is continuous and maps bounded sets into bounded sets.
A2. f is Fréchet differentiable on $H_0^1(\Omega)$.

Let $u_h \in S_h$ be a finite element approximate solution of (4.1) satisfying

$$(\nabla u_h, \nabla v_h)_{L^2} = (f(u_h), v_h)_{L^2} \qquad \text{for all } v_h \in S_h.$$

We used the library PROFIL (*cf.* [4]), which enables us to enclose the above u_h in very small intervals. We attempt to find the solution of (4.1) near $\breve{u}$ satisfying

$$\begin{cases} -\Delta \breve{u} &= f(u_h) \quad \text{in } \Omega, \\ \breve{u} &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

For this $\breve{u} \in H^2(\Omega) \bigcap H_0^1(\Omega)$, the relation $u_h = P_{h0}\breve{u}$ holds, as can be confirmed by a simple calculation. From (4.1) and (4.2), we have

$$\begin{cases} -\Delta(u - \breve{u}) &= f(u) - f(u_h) \quad \text{in } \Omega, \\ u - \breve{u} &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Defining $v_0 = \breve{u} - u_h$, we then have $v_0 \in S_h^{\perp}$, and we can write

$$\breve{u} = u_h + v_0 \qquad \text{for } u_h \in S_h \text{ and } v_0 \in S_h^{\perp}.$$

Similar to the discussion in Section 2, we obtain the following estimation:

$$\|v_0\|_{H_0^1} \leq \|\nabla u_h - \bar{\nabla} u_h\|_{L^2} + C_0 h \|\bar{\Delta} u_h + f(u_h)\|_{L^2}.$$

Note also that in this estimation we used the L^2-estimate of v_0:

$$\|v_0\|_{L^2} \leq C_0 h \|v_0\|_{H_0^1}.$$

Now, in order to verify solutions u of (4.1) near $\breve{u}$, writing $w = u - \breve{u}$, we can rewrite (4.3) as follows:

$$\begin{cases} -\Delta w &= f(u_h + v_0 + w) - f(u_h) \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Let $(-\Delta)^{-1}\psi$ be the solution of (2.2) for $\psi \in L^2(\Omega)$. Then

$$(-\Delta)^{-1} : L^2(\Omega) \longrightarrow H^2(\Omega) \cap H_0^1(\Omega)$$

is a bounded operator, and since the imbedding $H^2(\Omega) \hookrightarrow H_0^1(\Omega)$ is compact,

$$(-\Delta)^{-1} : L^2(\Omega) \longrightarrow H_0^1(\Omega)$$

is a compact operator. Thus using the compact map $F_0 : H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$ defined by

$$F_0(w) \equiv (-\Delta)^{-1}(f(u_h + v_0 + w) - f(u_h)), \tag{4.5}$$

we have the fixed point equation for w on $H_0^1(\Omega)$,

$$w = F_0(w). \tag{4.6}$$

Next, let $F_0'(-v_0)$ be the Fréchet derivative of F_0 at $-v_0$, and define $L \equiv I_0 - F_0'(-v_0)$. Moreover, with $\widetilde{L} \equiv (-\Delta)L : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$, by $F_0'(-v_0)u = (-\Delta)^{-1}f'(u_h)u$, we have

$$\widetilde{L}u = -\Delta u - f'(u_h)u$$

which is a strongly elliptic operator. Here $-\Delta$ is interpreted in a distributional sense. Note that, by restricting the domain of definition of $\widetilde{L}$ to $H^2(\Omega) \bigcap H_0^1(\Omega)$, we can regard the operator $\widetilde{L} : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ as $\widetilde{L} : H^2(\Omega) \bigcap H_0^1(\Omega) \longrightarrow L^2(\Omega)$.

Now we discuss the existence of $\widetilde{L}^{-1}$.

Since

$$\widetilde{L} = (-\Delta)(I_0 - F_0'(-v_0))$$

holds, $(-\Delta)^{-1}\widetilde{L}$ is a Fredholm operator of index 0. If the ESAV λ_* of $\widetilde{L}$ is known to be nonzero, then we have the estimate

$$\|u\|_{L^2} \leq \frac{1}{|\lambda_*|}\|\widetilde{L}u\|_{L^2} \qquad \text{for } u \in H^2(\Omega) \cap H_0^1(\Omega),$$

and therefore,

$$\widetilde{L}u = 0 \Longleftrightarrow u = 0,$$

which implies that $(-\Delta)^{-1}\widetilde{L}$ is an injection. Thus by the Fredholm alternative theorem, $(-\Delta)^{-1}\widetilde{L}$ has an inverse. Hence, the existence of the continuous operator $\widetilde{L}^{-1} : L^2(\Omega) \longrightarrow H^2(\Omega) \bigcap H_0^1(\Omega)$ is confirmed.

Now, define the operator $T_0 : H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$ as follows:

$$T_0(w) \equiv \widetilde{L}^{-1}(f(u_h + v_0 + w) - f(u_h) - f'(u_h)w). \tag{4.7}$$

This operator is derived by a Newton-like method for the equation $(I_0 - F_0)w = 0$. Then, since the imbedding $H^2(\Omega) \hookrightarrow H_0^1(\Omega)$ is compact, using assumptions **A1** and **A2**, T_0 becomes a compact operator on $H_0^1(\Omega)$, and we have

$$w = T_0(w) \Longleftrightarrow w = F_0(w).$$

4.2 Verification conditions

In this subsection we present a verification condition with uniqueness based upon [25].

Now, we intend to find a solution to (4.1) in a set W_0. Taking a strictly positive real number α, a candidate set W_0 is defined by

$$W_0 \equiv \{w \in H_0^1(\Omega) |\ \|w\|_{H_0^1} \leq \alpha\}. \tag{4.8}$$

By the method described below we choose strictly positive constants β and γ such that

$$\|T_0(0)\|_{H_0^1} \leq \beta, \tag{4.9}$$

$$\|T_0'(w_1)w_2\|_{H_0^1} \leq \gamma \qquad \text{for all } w_1, w_2 \in W_0. \tag{4.10}$$

We then define the set $K_0 \subset H_0^1(\Omega)$ by

$$K_0 \equiv \{v \in H_0^1(\Omega) \mid \|v\|_{H_0^1} \leq \beta + \gamma\}, \tag{4.11}$$

which includes the image of W_0 transformed by the linearized operator of T_0. The verification condition is described in the following theorem.

Theorem 5. *If $K_0 \subset W_0$ holds for a candidate set W_0 defined by (4.8), namely,*

$$\beta + \gamma \leq \alpha, \tag{4.12}$$

then there exists a solution to (4.1) in $\breve{u} + K_0$. Moreover, this solution is unique within the set $\breve{u} + W_0$.

Since the proof of this theorem is similar to that of Theorem 1, we describe the outline of it.

Outline of proof.

We define a scaling norm $\|\cdot\|_{W_0}$ by

$$\|x\|_{W_0} \equiv \frac{\|x\|_{H_0^1}}{\alpha} \qquad \text{for } x \in H_0^1(\Omega),$$

where α is the same one in (4.8). Using this norm we can prove that

$$T_0(W_0) \subset W_0$$

and

$$\|T_0(w_2) - T_0(w_1)\|_{W_0} \leq k\|w_2 - w_1\|_{W_0}$$

for some k s.t. $0 < k < 1$ and for all $w_1,\ w_2 \in W_0$. Then we can apply Banach's fixed point theorem to obtain the desired conclusion. ∎

4.3 Estimation of constants and algorithm

In this subsection we describe the manner in which the constants in (4.9) and (4.10) are obtained.

Constant β

Since

$$T_0(0) = L^{-1}(-\Delta)^{-1}(f(u_h + v_0) - f(u_h))$$

holds, we consider constants ζ_1 and η satisfying

$$\|T_0(0)\|_{H_0^1} \leq \zeta_1\|f(u_h + v_0) - f(u_h)\|_{L^2}, \tag{4.13}$$
$$\|f(u_h + v_0) - f(u_h)\|_{L^2} \leq \eta. \tag{4.14}$$

We then can choose β to be $\zeta_1\eta$.

As for ζ_1, we first calculate the constant ζ_0 such that

$$\|u\|_{L^2} \leq \zeta_0\|(-\Delta)Lu\|_{L^2} \qquad \text{for } u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{4.15}$$

This constant ζ_0 is obtained as the inverse of the ESAV of

$$(-\Delta)Lu = \lambda u. \tag{4.16}$$

Then we can calculate ζ_1 from ζ_0 as follows (cf.[17]). Assume that $L^\infty(\Omega) \ni q \equiv -f'(u_h)$ and that there exist constants $\underline{q}, \bar{q}$ such that

$$\underline{q} \leq q(x) \leq \bar{q} \quad (x \in \bar{\Omega}). \tag{4.17}$$

Then the constant ζ_1 is derived as follows:

$$\zeta_1 = \begin{cases} \sqrt{\zeta_0(1 - \underline{q}\zeta_0)} & \text{if } \underline{q}\zeta_0 \leq \frac{1}{2}, \\ \dfrac{1}{2\sqrt{\underline{q}}} & \text{otherwise.} \end{cases} \tag{4.18}$$

Since $f(\cdot) \in L^2(\Omega)$ is assumed to be bounded, η can be calculated using a simple estimation.

Constant γ

We consider a constant ζ_2 and a monotonically increasing function $Q : [0, \infty) \longrightarrow [0, \infty)$ such that for all $w_1, w_2 \in W_0$,

$$\|T_0'(w_1)w_2\|_{H_0^1} \leq \zeta_2 \|f'(u_h + v_0 + w_1)w_2 - f'(u_h)w_2\|_{L^2}, \tag{4.19}$$

$$\|f'(u_h + v_0 + w_1)w_2 - f'(u_h)w_2\|_{L^2} \leq Q(\|w_2\|_{H_0^1}). \tag{4.20}$$

Then since we have

$$\|T_0'(w_1)w_2\|_{H_0^1} \leq \zeta_2 Q(\|w_2\|_{H_0^1}) \qquad \text{for all } w_1, w_2 \in W_0,$$

we can choose γ as $\zeta_2 Q(\alpha)$.

As for ζ_2, since

$$(-\Delta)LT_0'(w_1)w_2 = f'(u_h + v_0 + w_1)w_2 - f'(u_h)w_2$$

holds for all $w_1, w_2 \in W_0$, we can choose $\zeta_2 = \zeta_1$. With regard to the monotonically increasing function Q, since $f' : H_0^1(\Omega) \longrightarrow \mathcal{L}(H_0^1(\Omega), L^2(\Omega))$ (the space of bounded and linear operators $H_0^1(\Omega) \longrightarrow L^2(\Omega)$) is a bounded and linear operator, we can construct Q as

$$Q(\|w_2\|_{H_0^1}) = C_{(u_h,v_0,\alpha)}\|w_2\|_{H_0^1},$$

where $C_{(u_h,v_0,\alpha)}$ is a positive constant independent of $w_2 \in W_0$.

Now, we describe an algorithm for finding a real number α which satisfies the verification condition (4.12),

$$\beta + \gamma \leq \alpha.$$

Since γ depends on α, we write γ as

$$\gamma = \gamma(\alpha).$$

If $f(u)$ in (4.1) is a polynomial in u, $\gamma(\alpha)$ is a polynomial in α. Therefore, in order to find α satisfying (4.12), we may solve the inequality for α. Here we present the following iteration method.

Algorithm
1. Fix a maximum iteration number.
2. Find a constant β satisfying (4.9).
3. Set $\alpha \leftarrow \beta$.
4. Find a constant $\gamma(\alpha)$ satisfying (4.10).
5. Check the verification condition (4.12); $\beta + \gamma(\alpha) \leq \alpha$.
   If the condition is satisfied, the verification has succeeded. If not,

$$\text{Set } \alpha \leftarrow (1+\delta)\alpha \ (0 < \delta \ll 1), \tag{4.21}$$

where δ $(0 < \delta \ll 1)$ represents an inflation parameter (*cf.*[19],[23] etc.). Then increase the iteration number by 1 and return to step 4.
6. If the maximum iteration number is exceeded, without (4.12) being satisfied, the verification has failed.

5 Numerical examples

First, we give two examples whose eigenvalues were enclosed using the method in Section 2. Specifically, we have verified the eigenvalues with the smallest absolute values (ESAV) with the technique described in Section 3. We set the following conditions:

Ω is the rectangular domain $(0,1) \times (0,1) \subset R^2$, and the interval (0,1) was partitioned into 20 pieces ($h = \frac{1}{20}$). The basis in S_h consists of continuous, piecewise biquadratic polynomials on Ω. ($M = \mathrm{dim} S_h = 1521$) The inflation parameter δ in (2.40) is set to 0.0001. Then the constants appearing previously can be taken as $C_1 = \frac{1}{2\pi}$ and $C_2 = 1$ (*cf.*[14]).

In the calculations, interval arithmetic is used to avoid the effects of rounding errors in the floating-point computations. The computations were carried out on a Sun Enterprise 450 using the interval library PROFIL coded by Knüppel of the Technical University of Hamburg-Harburg ([4]). PROFIL is implemented as a portable C++ class fast interval library and supports two interval solvers proposed by Rump ([19]).

Example 1:

We consider the following self-adjoint eigenvalue problem:

$$\begin{cases} -\Delta u - 2u_{h1}u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where u_{h1} is a finite element solution of the following so-called Emden equation:

$$\begin{cases} -\Delta u &= u^2 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

We calculated a finite element approximate solution $(\widehat{u}_h, \widehat{\lambda}_h)$ satisfying (2.6) using the interval Newton method. The verified results for the ESAV are given in Table 1. These demonstrate that the ESAV exists in the interval

$$\widehat{\lambda}_h + [-(Y_{M+2} + Z_{M+2}), Y_{M+2} + Z_{M+2}],$$

and that it is unique in the interval

$$\widehat{\lambda}_h + [-W_{M+2}, W_{M+2}].$$

	Example 1	Example 2
$\widehat{\lambda}_h$	$[-20.70639871^{6851}_{7881}]$	$[-16.610392^{2420}_{3224}]$
$\Vert\widehat{u}_h\Vert_{L^\infty}$	2.47738622	2.30812688
$\Vert v_1\Vert_{L^2}$	0.00013965	0.00013715
Iteration number	9	9
$\max_{1\le i\le M}(Y_i+Z_i)$	0.00296313	0.00320281
$\max_{1\le i\le M} W_i$	0.00296339	0.00320311
$Y_{M+1}+Z_{M+1}$	0.00111568	0.00104195
W_{M+1}	0.00111569	0.00104204
$Y_{M+2}+Z_{M+2}$	0.06131755	0.05976070
W_{M+2}	0.06132296	0.05976615

Table 1: Results of verification

Example 2:

We consider the following self-adjoint eigenvalue problem:

$$\begin{cases} -\Delta u+\nu(3u_{h2}^2-2(a+1)u_{h2}+a)u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

where ν and a are positive constants, and u_{h2} is a finite element solution of the following so-called Allen-Cahn equation:

$$\begin{cases} -\Delta u &= \nu u(u-a)(1-u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

It is known that this equation has two solution branches with respect to the parameter $\nu>0$ (*cf.*[23]). Here we considered the case in which u_{h2} is an approximate lower branch solution for $\nu=150$, $a=0.01$. As in Example 1, we calculated a finite element approximate solution $(\widehat{u}_h,\widehat{\lambda}_h)$ satisfying (2.6) using the interval Newton method. The verification results are given in Table 1. They also demonstrate that the ESAV exists in the interval

$$\widehat{\lambda}_h+[-(Y_{M+2}+Z_{M+2}),Y_{M+2}+Z_{M+2}],$$

and that it is unique in the interval

$$\widehat{\lambda}_h+[-W_{M+2},W_{M+2}].$$

Next we give two examples, the Emden equation and the Allen-Cahn equation, which we have studied using the method in Section 4. In these examples, Ω, the number of partitions for the interval (0,1) and the basis in S_h are the same as previous two examples, except for the inflation parameter δ in (4.21), 0.01.

Example 3:

We consider to find a nontrivial solution of the following so-called Emden equation:

$$\begin{cases} -\Delta u &= u^2 \quad \text{in } \Omega, \\ \quad u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

The associated eigenvalue problem for the estimation of the norm of the inverse of the linearized operator here is (5.1).

By the method described in Section 3, we have verified that the interval

$$[-20.6, 55.5]$$

does not contain any eigenvalues of (5.1), and we can thus take the bound of the ESAV as 20.6. Thus, it is seen that the constants ζ_0 and ζ_1 in (4.13) and (4.15) can be taken as

$$\zeta_0 = 0.0485438,$$

$$\zeta_1 = 0.225566.$$

We can take the constant η in (4.14) as

$$\eta = 0.0632076.$$

Also, the monotonically increasing function Q satisfying (4.20) can be chosen as

$$Q(x) = \left(\frac{2}{\pi^2}\alpha + \frac{2}{\pi}C_3\|v_0\|_{H_0^1}\right)x,$$

where C_3 is a constant determined by

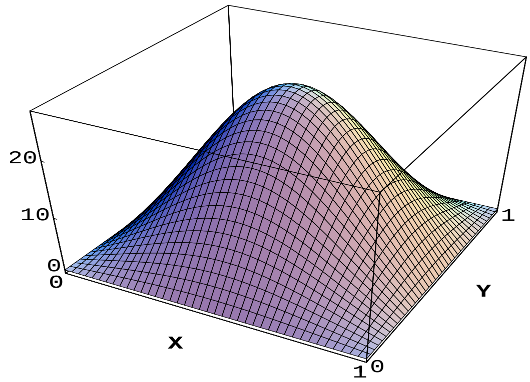

Figure 2: Approximate solution u_{h1}

$$C_3 = \left(\frac{1}{40\pi}\right)^{\frac{7}{16}} \left\{\frac{9}{2\pi 8^{\frac{1}{18}}}\sqrt{\sin\frac{\pi}{9}}\right\}^{\frac{9}{16}}.$$

See Appendix for the details of the method to calculate above η, $Q(x)$ and C_3. We completed the verification algorithm presented in Section 4.3 with 2 iterations.

The constant α in the set W_0 defined in (4.8) and the constant $\beta+\gamma$ in the set K_0 defined in (4.11) were determined as

$$\alpha = 0.014399992,$$

$$\beta + \gamma = 0.014295359.$$

That is, there exists a solution u in (5.5) satisfying

$$\|u - \check{u}\|_{H_0^1} \leq 0.014295359.$$

Moreover, it is unique in

$$\|u - \check{u}\|_{H_0^1} \leq 0.014399992.$$

Figure 2 displays the shape of an approximate solution of (5.5).

Example 4:

We next consider to find a nontrivial solution of the following so-called Allen-Cahn equation:

$$\begin{cases} -\Delta u &= \nu u(u-a)(1-u) \quad \text{in } \Omega, \\ u &= 0 \quad\quad\quad\quad\quad\quad\quad \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

We have verified the existence and uniqueness of two exact solutions on the upper and lower branch in the case that $\nu = 150$ and $a = 0.01$. Then the associated eigenvalue problem for the estimation of the norm of the inverse of the linearized operator is (5.3).

The upper branch solution

With the method described in Section 3, we have verified that the interval

$$[-50.0, 45.0]$$

does not contain any eigenvalues of (5.3), and thus we can take the bound of the ESAV as 45.0. Then it follows that the constants ζ_0 and ζ_1 in (4.13) and (4.15) can be taken as

$$\zeta_0 = 0.02222222,$$

$$\zeta_1 = 0.149173.$$

We can choose the constant η in (4.14) as

$$\eta = 0.0107114,$$

and the monotonically increasing function Q satisfying (4.20) can be chosen as

$$\begin{aligned} Q(x) \;=\; \nu\Big\{ &\|6u_{h2} - 2(a+1)\|_\infty (C_3\|v_0\|_{H_0^1} + C_4\alpha)C_4 \\ &+3C_4(C_5\|v_0\|_{H_0^1} + C_6\alpha)^2\Big\}x, \end{aligned}$$

where C_4, C_5 and C_6 are constants determined by

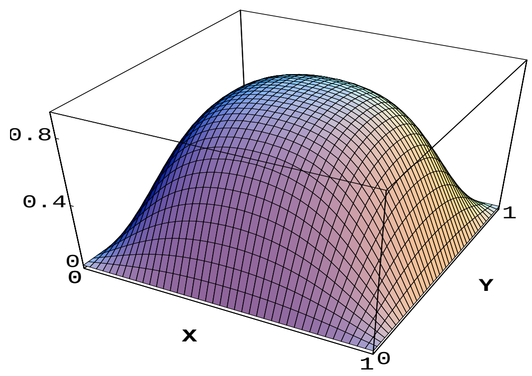

Figure 3: Approximate solution u_{h2} (upper branch)

$$C_4 = \frac{1}{\pi},$$

$$C_5 = \left(\frac{1}{40\pi}\right)^{\frac{1}{8}} \left\{ \frac{7}{2\pi 6^{\frac{1}{14}}} \sqrt{\sin\frac{\pi}{7}} \right\}^{\frac{7}{8}},$$

$$C_6 = \frac{2}{3^{\frac{1}{8}} 2^{\frac{1}{4}} \pi}.$$

For the calculations of above η, $Q(x)$, C_4, C_5 and C_6, see Appendix.

Verification by the algorithm presented in Section 4.3 was completed with 6 iterations. The constant α in the set W_0 defined in (4.8) and the constant $\beta+\gamma$ in the set K_0 defined in (4.11) were determined as

$$\alpha = 0.00167936,$$

$$\beta + \gamma = 0.00167052.$$

That is, there exists a solution u in (5.6) satisfying

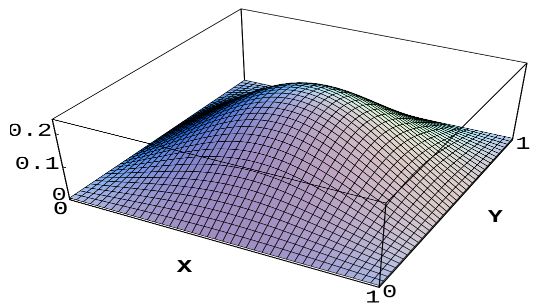

Figure 4: Approximate solution u_{h2} (lower branch)

$$\|u - \check{u}\|_{H_0^1} \leq 0.00167052.$$

Moreover, it is unique in

$$\|u - \check{u}\|_{H_0^1} \leq 0.00167936.$$

The lower branch solution

We verified that the interval

$$[-16.2, 18.0]$$

does not contain any eigenvalues of (5.3). This implies that the bound of the ESAV can be taken as 16.2. Similarly, as before, we obtain

$$\zeta_0 = 0.0617284,$$

$$\zeta_1 = 0.24852,$$

$$\eta = 0.00041168.$$

The monotonically increasing function Q satisfying (4.20) is the same here as in the upper branch.

We completed the verification in this case with 2 iterations. The constant α in the set W_0 defined in (4.8) and the constant $\beta + \gamma$ in the set K_0 defined in (4.11) were determined as

$$\alpha = 0.00010333,$$

$$\beta + \gamma = 0.00010266.$$

That is, there exists a solution u in (5.6) satisfying

$$\|u - \check{u}\|_{H_0^1} \leq 0.00010266.$$

Moreover, it is unique in

$$\|u - \check{u}\|_{H_0^1} \leq 0.00010333.$$

Figures 3 and 4 display contours of the approximate upper and lower branch solutions of (5.6), respectively.

6 Conclusions

It is very interesting and important problem to verify the eigenvalues with guaranteed error bounds in mathematics as well as physics and engineering. In the existing verified computational approaches, the methods in [2] and in [15, 16] are able to guarantee the index of the enclosed eigenvalue. On the other hand, our method has an advantage that it is applicable to non-selfadjoint eigenvalue problems while other methods, in [2] and [15, 16], seem to be difficult to deal with such kind of problems. In this paper, we applied the verification method for the nonlinear elliptic boundary value problems to the eigenvalue enclosing. This method assures the simplicity of the enclosed eigenvalue. Moreover, by estimating the norm of the inverse linearized operator for nonlinear elliptic problems using the ESAV, we proposed a new verification method for the solutions of that problems incorporating with the approach in [17].

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Appendix

In this appendix, we describe the method to compute the constant η in (4.14) and the monotonically increasing function Q satisfying (4.20). In order to obtain better constant η and function Q, we used the best imbedding constatns by G.Talenti (cf. [22]). Therefore we first describe the imbedding constants.

If the domain $\Omega \subset R^2$ has a cone property, by the Sobolev imbedding theorems (cf. [1]), there exists the imbedding

$$H^1(\Omega) \hookrightarrow L^p(\Omega), \qquad 2 \le p < \infty,$$

that is, we have

$$u \in H^1(\Omega) \;\Rightarrow\; u \in L^p(\Omega).$$

The following theorem is the main result in [22].

Theorem

Let u be any real (or complex) valued function, defined on the whole n-dimensional Euclidean space R^n, sufficiently smooth and decaying fast enough at infinity. Moreover let p be any number such that $1 < p < n$. Then

$$(1) \qquad \left\{\int_{R^n} |u|^q \, dx\right\}^{1/q} \le C \left\{\int_{R^n} |Du|^p \, dx\right\}^{1/p},$$

where $|Du|$ is the length of the gradient Du of u, $q = np/(n-p)$ and

$$(2) \qquad C = \pi^{-1/2} n^{-1/p} \left(\frac{p-1}{n-p}\right)^{1-1/p} \left\{\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right\}^{1/n}.$$

The equality sign holds in (1) if u has the form

$$(3) \qquad u(x) = [a + b|x|^{p/(p-1)}]^{1-n/p},$$

where $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $a,\ b$ are positive constatns.

In the above theorem $\Gamma(n)$ means the gamma function.

Now, we consider the case $n = 2$. In the present case $\Omega = (0,1) \times (0,1)$ we can extend $u \in H_0^1(\Omega)$ to the whole space R^2 as follows:

$$(4) \qquad u(x) = \begin{cases} u(x), & (x \in \Omega) \\ 0. & (x \notin \Omega) \end{cases}$$

By (4) using Hölder's inequality we have

$$\begin{aligned} \left(\int_{R^2} |Du|^p \, dx \right)^{1/p} &= \left(\int_{\Omega} |Du|^p \, dx \right)^{1/p} \\ &\leq \left\{ \left(\int_{\Omega} (|Du|^p)^{2/p} \, dx \right)^{p/2} \left(\int_{\Omega} 1^{2/(2-p)} \, dx \right)^{(2-p)/2} \right\}^{1/p} \\ &\leq |\Omega|^{(2-p)/2p} \|\nabla u\|_{L^2}. \end{aligned}$$

Therefore using the constant C in (1) we have

$$\|u\|_{L^q} \leq C|\Omega|^{(2-p)/2p} \|\nabla u\|_{L^2}.$$

If we write $q = 2k$, p is then written as $p = 2k/(k+1)$, and $1 < p < n = 2$ for $k > 1$. Since Γ function have the properties

$$\Gamma(x+1) = x\Gamma(x), \ \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

using these properties, above constant C is written as

$$(5) \qquad C = \frac{k}{2\pi}(k-1)^{-1/2k} \left(\sin \frac{\pi}{k} \right)^{1/2}.$$

Considering this constant C is depend on k, we write C as $C(k)$. Then the following estimates hold:

$$(6) \qquad \|u^k\|_{L^2} = \|u\|^k_{L^{2k}} \leq (C(k))^k |\Omega|^{1/2} \|u\|^k_{H^1_0}.$$

If $\|u\|_{L^2}$ is small, we can sometimes obtain better estimates by using $\|u\|_{L^2}$ together with (6) as described below.

By Schwarz' inequality we have

$$(7) \qquad \|u^k\|_{L^2} \leq \sqrt{\|u\|_{L^2} \|u^{2k-1}\|_{L^2}}.$$

Then, by applying the estimates (6) to $\|u^{2k-1}\|_{L^2}$ in (7), we have

$$(8) \qquad \|u^k\|_{L^2} \leq (C(2k-1))^{k-1/2}|\Omega|^{1/4}\|u\|_{L^2}^{1/2}\|u\|_{H_0^1}^{k-1/2}.$$

Moreover, by applying the estimates (8) to $\|u^{2k-1}\|_{L^2}$ in (7), we obtain the estimates

$$(9) \qquad \|u^k\|_{L^2} \leq (C(4k-3))^{(4k-3)/4}|\Omega|^{1/8}\|u\|_{L^2}^{3/4}\|u\|_{H_0^1}^{(4k-3)/4}.$$

Repeating the above technique l times, we have for $k' \equiv k - (1 - \frac{1}{2^l})$,

$$(10) \qquad \|u^k\|_{L^2} \leq (C(2^l k'))^{k'}|\Omega|^{1/2^{(l+1)}}\|u\|_{L^2}^{k-k'}\|u\|_{H_0^1}^{k'}.$$

Now, using above estimates, we calculate the constant η in (4.14) and the monotonically increasing function Q satisfying (4.20).

Constant η

For Example 3, we have

$$\begin{aligned} \|f(u_{h1}+v_0) - f(u_{h1})\|_{L^2} &= \|2u_{h1}v_0 + v_0^2\|_{L^2} \\ &\leq 2\|u_{h1}\|_\infty\|v_0\|_{L^2} + \|v_0^2\|_{L^2}, \end{aligned}$$

and for Example 4, we have

$$\begin{aligned} & \|f(u_{h2}+v_0) - f(u_{h2})\|_{L^2} \\ = \ & \nu\|v_0(-3u_{h2}^2 - 3u_{h2}v_0 - v_0^2 + 2(a+1)u_{h2} + (a+1)v_0 - a)\|_{L^2}. \end{aligned}$$

From the above, we need to estimate $\|v_0^2\|_{L^2}$ and $\|v_0^3\|_{L^2}$. In order to obtain better estimates for these norms, we used the estimates (10). Since we have, by Schwarz' inequality,

$$\|v_0^2\|_{L^2} \leq \sqrt{\|v_0\|_{L^2}\|v_0^3\|_{L^2}},$$

the estimates of $\|v_0^3\|_{L^2}$ plays an important role. As we could obtain better estimate for it in case $l = 2$ and $k = 3$ in (10), we calculated the constant η by using it.

Function Q

For Example 3, we have for $w_1,\ w_2 \in W_0$

$$\begin{aligned} & \|f'(u_{h1}+v_0+w_1)w_2 - f'(u_{h1})w_2\|_{L^2} \\ = \; & 2\|(v_0+w_1)w_2\|_{L^2} \\ \le \; & 2(\|v_0\|_{L^4} + \|w_1\|_{L^4})\|w_2\|_{L^4} \end{aligned}$$

and for Example 4, we have

$$\begin{aligned} & \|f'(u_{h2}+v_0+w_1)w_2 - f'(u_{h2})w_2\|_{L^2} \\ = \; & \nu\|(v_0+w_1)\{6u_{h2}+3(v_0+w_1)-2(a+1)\}w_2\|_{L^2} \\ \le \; & \nu\|6u_{h2}-2(a+1)\|_\infty(\|v_0\|_{L^4}+\|w_1\|_{L^4})\|w_2\|_{L^4} + 3\nu(\|v_0\|_{L^8}+\|w_1\|_{L^8})^2\|w_2\|_{L^4}. \end{aligned}$$

Therefore we need the estimates for $\|v_0\|_{L^4},\ \|v_0\|_{L^8}$ and $\|w\|_{L^4}, \|w\|_{L^8}$ for $w \in W_0$. We used (6) for $\|w\|_{L^4}, \|w\|_{L^8}$. Since $\|v_0\|_{L^4}$ is estimated by

$$\|v_0\|_{L^4} \le \|v_0\|_{L^2}^{1/4}\|v_0^3\|_{L^2}^{1/4},$$

$\|v_0^3\|_{L^2}$ is essential for the estimates of $\|v_0\|_{L^4}$ and we used the same estimate above, i.e., $l = 2$ and $k = 3$ in (10). For $\|v_0\|_{L^8}$, since we have, by Schwarz' inequality,

$$\|v_0\|_{L^8} \le \|v_0\|_{L^2}^{1/8}\|v_0^7\|_{L^2}^{1/8},$$

the estimates of $\|v_0^7\|_{L^2}$ is important. As we could obtain better estimate for it in case $k = 7$ in (6), we constitute the function Q by using it.

References

[1] R.A.Adams, Sobolev spaces, Academic Press, New York (1975).

[2] H.Behnke and F.Goerisch, Inclusions for Eigenvalues of Selfadjoint Problems, In: J.Herzberger(eds.), Topics in Validated Computations-Studies in Computational Mathematics, Elsevier, Amsterdam (1994).

[3] P.Grisvard, Elliptic problems in nonsmooth domains, Pitman Monographs and Surveys in Pure and Applied Mathematics **24** (1985), London.

[4] O.Knüppel, PROFIL/BIAS - A fast interval library, Computing **53** (1994), 277-288.

[5] M.Křížek, P.Neittaanmäki , Finite Element Approximation of Variational Problems and Applications, Longman Scientific and Technical, Harlow (1990).

[6] G.Mayer, Result verification for eigenvectors and eigenvalues, In: J.Herzberger(eds.), Topics in Validated Computations-Studies in Computational Mathematics, Elsevier, Amsterdam (1994).

[7] K.Nagatou, N.Yamamoto and M.T.Nakao, An approach to the numerical verification of solutions for nonlinear elliptic problems with local uniqueness, to appear in Numerical Functional Analysis and Optimization.

[8] K.Nagatou, A numerical method to verify the elliptic eigenvalue problems including a uniqueness property, submitted to *Computing* for publication.

[9] M.T.Nakao, A numerical approach to the proof of existence of solutions for elliptic problems, Japan Journal of Applied Mathematics **5** (1988), 313 - 332.

[10] M.T.Nakao, A numerical approach to the proof of existence of solutions for elliptic problems II, Japan Journal of Applied Mathematics **7** (1990), 477-488.

[11] M.T.Nakao, A numerical verification method for the existence of weak solutions for nonlinear boundary value problems, Journal of Mathematical Analysis and Applications **164** (1992), 489-507.

[12] M.T.Nakao, Solving Nonlinear Elliptic Problem with Result Verification Using an H^{-1} Type Residual Iteration, Computing, Suppl. **9** (1993), 161-173.

[13] M.T.Nakao and N.Yamamoto, Numerical verification of solutions for nonlinear elliptic problems using an L^∞ residual method, J.Math.Anal.Appl. **217** (1998), 246-262.

[14] M.T.Nakao, N.Yamamoto and K.Nagatou, Numerical Verifications for eigenvalues of second-order elliptic operators, to appear in Japan Journal of Industrial and Applied Mathematics.

[15] M.Plum, Eigenvalue inclusions for second-order ordinary differential operators by a numerical homotopy method, J.Appl.Math.Phys.(ZAMP) **41** (1990), 205-226.

[16] M.Plum, Bounds for eigenvalues of second-order elliptic differential operators, J.Appl.Math.Phys.(ZAMP) **42** (1991), 848-863.

[17] M.Plum, Explicit H_2-estimates and pointwise bounds for solutions of second-order elliptic boundary value problems, Journal of Mathematical Analysis and Applications **165** (1992), 36-61.

[18] M.Plum, Existence and enclosure results for continua of solutions of parameter-dependent nonlinear boundary value problems, Journal of Computational and Applied Mathematics **60**, (1995), 187-200.

[19] S.M.Rump, Solving algebraic problems with high accuracy, In: U.Kulisch, W.L.Miranker(eds.), A new approach to scientific computation, Academic Press, New York (1983).

[20] S.M.Rump, New Results on Verified Inclusions, Lecture Notes in Computer Science **235** (1986), 31-69.

[21] S.M.Rump, On the Solution of Interval Linear Systems, Computing **47** (1992), 337-353.

[22] G.Talenti, Best constant in Sobolev inequality, Ann. Math. Pura Appl. **110**, (1976) 353-372.

[23] Y.Watanabe and M.T.Nakao, Numerical verifications of solutions for nonlinear elliptic equations, Japan Journal of Industrial and Applied Mathematics **10** (1993), 165-178.

[24] N.Yamamoto and M.T.Nakao, Numerical verifications for solutions to elliptic equations using residual iterations with a high order finite element, Journal of Computational and Applied Mathematics **60** (1995), 271-279.

[25] N.Yamamoto, A numerical verification method for solutions of boundary value problems with local uniqueness by Banach's fixed point theorem, SIAM Journal on Numerical Analysis **35** (1998), 2004-2013.

[26] E.Zeidler, Nonlinear Functional Analysis and its Applications I, Springer, New York (1986).