MAXIMIZATION OF MULTIVARIABLE FUNCTIONS USING INTERVAL ANALYSIS

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I. INTRODUCTION

A number of techniques have been proposed for nonlinear optimization problems. Some of them are conjugate gradient method, simplex method, variable metric method and random search method [1].

However, if the objective function is multimodal, we have few methods for finding the global maximum or minimum [2].

Interval analysis is very effective for this global optimization problems[3],[4],[5]. The simple way to apply this method is to divide the original domain into subregions, and to delete subregions that can not have the global maximum [6]. This algorithm is not very efficient since a great many of subregions remain without being discarded.

In this paper we describe an interval method to compute the global maximum value of the multivariable function over the hyperrectangle.

The interval Newton method is used for finding the stationary points in the domain and on the boundary. On the boundary one or more variables are fixed as constants, so that the dimension of the Hessian matrix decreases.

The constrained optimization with equality or inequality condition can be solved by the Lagrange multiplier method.

Our interval arithmetic system is written with FORTRAN 77 and the assembly language. The upper and the lower bounds of the number can be calculated at an arbitrary digit.

II. UNCONSTRAINED MAXIMIZATION

II-1. One-Dimensional Case

We consider how to compute the greatest value of the function f(x) on a closed interval [a, b], where f is supposed to belong to C^2. The global maximum is obtained by computing the maximum of relative maxima in (a, b) and function values at the two end points.

The interval Newton algorithm is used to obtain the stationary values of f(x) by solving the equation f'(x)=0 [7]. It is given as

$$(1)\quad \left.\begin{aligned} N(X_p) &= m(X_p) - \frac{F'(m(X_p))}{F''(X_p)}, \\ X_{p+1} &= X_p \cap N(X_p), \end{aligned}\right\} \quad (p=0,1,\ldots)$$

where capital letters X, F', F'' denote interval extensions of x, f', f'' respectively and $m(X_p)$ is the midpoint of X_p. The maximization algorithm for one-variable function is as follows.

<u>Step 1</u>: Compute $f^* = \max[f(a), f(b)]$ and the point(s) x^* at which $f(x^*) = f^*$.

<u>Step 2</u>: Divide the original interval A=[a, b] into two subintervals at the midpoint m=(a+b)/2.

<u>Step 3</u>: If the widths of the undiscarded interval(s) are sufficiently small, then stop. Otherwise pick out an interval (let it be A_i), and calculate $F'(A_i)$ and $F''(A_i)$.

<u>Step 4</u>: If $F'(A_i) \not\ni 0$, then there is no stationary point in A_i. So delete A_i and go to Step 3. Otherwise go to Step 5.

<u>Step 5</u>: If $F''(A_i) > 0$, then the stationary value is a relative minimum

and A_i can be discarded. If $F''(A_i)<0$, compute this relative maximum by use of the interval Newton method. Replace f* and x* if the obtained relative maximum is greater than f*. If $F''(A_i)\ni 0$, divide A_i into two at its midpoint and go to Step 3.

II-2. Multi-Dimensional Case

Compute the maximum of multimodal function $f(x_1,x_2,\dots,x_k)$ in C^2 over a k-dimensional box $\mathbf{A}=[a_1,b_1]\times\dots\times[a_k,b_k]$. To seek the maximum of f, it is necessary to compute the values of relative maxima in $\mathbf{A}$ and maximum values on the boundary of $\mathbf{A}$. As compared with the one-dimensional case, we have the following difficulty.

[i] In the one-dimensional case we can distinguish relative maximum from relative minimum. In the multi-dimensional case the sign of the Hessian can not distinguish relative maximum from minimum or saddle point.

[ii] In the one-dimensional case the boundary consists of only two end points. In the multi-dimensional case the boundary becomes the lower-dimensional region on which the maximum must be sought.

As concerns [i] we calculate each stationary value, and replace the maximum value so far obtained with this value if it is larger.

Concerning [ii] we apply the interval Newton method in various dimensions (from 1 to k-1) to seek the stationary value on the boundary. For example if the original domain is four-dimensional, its boundary becomes three-dimensional on which three-dimensional Newton method is applied. Moreover the boundary of this three-dimensional region becomes two-dimensional, and so on. It should be noted that even if a region is deleted since no global maximum exists in it, we must save its boundary if it has a common boundary with the original domain.

The multi-dimensional interval Newton method corresponds to equation (1) is given as

$$(2)\qquad \left.\begin{aligned} \mathbf{N}(\mathbf{X}_n) &= \mathbf{m}(\mathbf{X}_n) - J^{-1}(\mathbf{X}_n)\nabla F(\mathbf{m}(\mathbf{X}_n)), \\ \mathbf{X}_{n+1} &= \mathbf{X}_n \cap \mathbf{N}(\mathbf{X}_n), \end{aligned}\right\}\ (n=0,1,\dots)$$

$$(3)\qquad \nabla F = \left(\frac{\partial F}{\partial X_1}, \frac{\partial F}{\partial X_2}, \dots, \frac{\partial F}{\partial X_k}\right)^T,$$

$$(4) \qquad J = \begin{bmatrix} \frac{\partial^2 F}{\partial X_1^2} & \frac{\partial^2 F}{\partial X_1 \partial X_2} & \cdots\cdots\cdots & \frac{\partial^2 F}{\partial X_1 \partial X_k} \\ \frac{\partial^2 F}{\partial X_2 \partial X_1} & \frac{\partial^2 F}{\partial X_2^2} & \cdots\cdots\cdots & \frac{\partial^2 F}{\partial X_2 \partial X_k} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 F}{\partial X_k \partial X_1} & \frac{\partial^2 F}{\partial X_k \partial X_2} & \cdots\cdots\cdots & \frac{\partial^2 F}{\partial X_k^2} \end{bmatrix}.$$

To get the Newton sequence (2), we set

$$(5) \qquad H_n = -J^{-1} \nabla F$$

and solve the linear equation

$$(6) \qquad J H_n = -\nabla F$$

with Gauss' elimination method.

On the boundary $x_1 = a_1$ (6) becomes the following (k-1)-dimensional equation since the elements differentiated with respect to X_1 vanish.

$$(7) \qquad \begin{bmatrix} 0 & 0 & \cdots\cdots\cdots & 0 \\ 0 & \frac{\partial^2 F}{\partial X_2^2} & \cdots\cdots\cdots & \frac{\partial^2 F}{\partial X_2 \partial X_k} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{\partial^2 F}{\partial X_k \partial X_2} & \cdots\cdots\cdots & \frac{\partial^2 F}{\partial X_k^2} \end{bmatrix} \begin{bmatrix} 0 \\ H_2 \\ \vdots \\ H_k \end{bmatrix} = - \begin{bmatrix} 0 \\ \frac{\partial F}{\partial X_2} \\ \vdots \\ \frac{\partial F}{\partial X_k} \end{bmatrix}.$$

The maximization algorithm for multi-variable function is as follows :

Step 1: Let f^* be the maximum value of f at the 2^k vertices.

Step 2: Divide **A** at the midpoint of the maximum side to generate two regions A_1 and A_2.

Step 3: If the widths of the undiscarded regions are sufficiently small, then stop. Otherwise take out a region A_i, and evaluate $\nabla F(A_i)$, $J(A_i)$ and $D = \det(J)$.

Step 4: If $0 \notin \nabla F(A_i)$, there is no stationary point in A_i. So discard A_i and go to Step 3. However, if A_i has a common boundary with A, then this common boundary should be saved as a (degenerate) region.

If $0 \in \nabla F(A_i)$, go to Step 5.

Step 5: If $0 \in D$, divide A_i at the midpoint of its maximum side and go to Step 3. Otherwise apply the interval Newton method to find the stationary value. If the obtained value is greater than f^*, f^* is replaced with it. Then go to Step 3. If A_i has a common boundary with the original domain A, the common boundary should be saved whether A_i has a stationary point or not.

III. CONSTRAINED MAXIMIZATION

We consider to find the value of $\mathbf{x}$ that maximizes

$$f(\mathbf{x}) = f(x_1, x_2, \ldots, x_k) \tag{8}$$

in the bounded region $[a_1, b_1] \times \ldots \times [a_k, b_k]$ subject to the r equality constraints

$$g_j(\mathbf{x}) = 0 \quad (j=1,\ldots,r). \tag{9}$$

To solve this problem we use the Lagrange function

$$L(\mathbf{x}, \mathbf{p}) = f(\mathbf{x}) + \sum_{j=1}^{r} p_j g_j(\mathbf{x}), \tag{10}$$

where $\mathbf{p} = (p_1, \ldots, p_r)^T$ is a Lagrange-multiplier [8]. The necessary condition at a maximum of the function is

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{r} p_j \frac{\partial g_j(\mathbf{x})}{\partial x_i} = 0 \quad (i=1,2,\ldots,k), \tag{11}$$

$$(12) \qquad \frac{\partial L}{\partial p_j} = g_j(\mathbf{x}) = 0 \quad (j=1,2,\ldots,r).$$

We can apply the interval Newton method in section II-2 to solve these nonlinear simultaneous equations. The constrained global maximum is obtained among the stationaly values of L.

Inequality constraints can be converted to the equality constraints by introducing an extra slack variable. The expression

$$(13) \qquad g_j(\mathbf{x}) \leqq 0 \quad (j=1,\ldots,r)$$

can be written as

$$(14) \qquad g_j(\mathbf{x}) + x_{k+j}^{\ 2} = 0 \quad (j=1,\ldots,r).$$

Then Lagrange function L is written as

$$(15) \qquad L(\mathbf{x},\mathbf{p}) = f(\mathbf{x}) + \sum_{j=1}^{r} p_j(g_j + x_{k+j}^{\ 2}).$$

The stationary points can be computed by the interval Newton method as before. The case that both equality and inequality constraints are contained can be treated similarly.

IV. NUMERICAL EXAMPLES

Several numerical examples have been computed by the method described above.

The calculations were done with HITAC M200H (corresponds to IBM 3083JX) of the Educational Center for Information Processing of Kyoto University.

Example 1: Three Hump Camel-Back Function.

$$(16) \qquad f(x_1,x_2) = -2x_1^{\ 2} + 1.05x_1^{\ 4} - \tfrac{1}{6}x_1^{\ 6} - x_1x_2 - x_2^{\ 2}.$$

This function is known to have three maxima and two saddle points in the domain $-5 \leq x_1 \leq 5$, $-4 \leq x_2 \leq 4$.

The computed result is:

X_1=[-0.49630 83675 31816 60D-22, 0.49630 83675 31816 60D-22],
X_2=[-0.70409 33880 50906 57D-22, 0.79409 33880 50906 57D-22],
F(max)=[-0.15173 43493 40171 69D-43, 0.19705 75965 45677 71D-45].

Example 2: Six Hump Camel-Back Function.

This function is known to have six maxima, two minima and seven saddle points in the domain $-5 \leq x_1 \leq 5$, $-4 \leq x_2 \leq 4$.

(17) $$f(x_1,x_2)=-4x_1^2+2.1x_1^4 - \frac{1}{3}x_1^6 -x_1x_2 +4x_2^2 -4x_2^4.$$

Global maximum is obtained at the following two points. They are symmetric with respect to the origin.

X_1=[0.08984 20131 00318 030, 0.08984 20131 00318 099],
X_2=[-0.71265 64030 20739 90 , -0.71265 64030 20739 40],
F(max)=[1.03162 84534 89873 6 , 1.03162 84534 89881 2].

X_1=[-0.08984 20131 00318 099, -0.08984 20131 00318 030],
X_2=[0.71265 64030 20739 40 , 0.71265 64030 20739 90],
F(max)=[1.03162 84534 89873 6 , 1.03162 84534 89881 2].

Example 3: Five-variable Function [9].

(18) $$f(x_1,x_2,\ldots,x_5)=f_1(x_1)f_2(x_2)f_3(x_3)f_4(x_4)f_5(x_5),$$

where

$$f_1(x_1)=x_1(x_1+13)(x_1-15)*0.01,$$
$$f_2(x_2)=(x_2+15)(x_2+1)(x_2-8)*0.01,$$
$$f_3(x_3)=(x_3+9)(x_3-2)(x_3-9)*0.01,$$
$$f_4(x_4)=(x_4+11)(x_4+5)(x_4-9)*0.01,$$
$$f_5(x_5)=(x_5+9)(x_5-9)(x_5-10)*0.01.$$

Case (i): $-10 \leq x_i \leq 10$, $(i=1,2,\ldots,5)$.

This function has 2^4 maxima and 2^4 minima in this domain.

The computed result is:

X_1=[8.75644 07330 07731 2, 8.75644 07330 07731 2],
X_2=[-9.35828 66332 94911 5, -9.35828 66332 94911 5],
X_3=[-4.57207 78818 33903 6, -4.57207 78818 33903 6],

X_4=[3.59212 96115 43726 2, 3.59212 96115 43726 2],
X_5=[-2.84008 63924 84045 8, -2.84008 63924 84043 4],
F(max)=[24416.03065 50573 60 , 24416.03065 50574 10].

Case (ii): $-10 \leq x_1 \leq 8$, $-10 \leq x_i \leq 10$, (i=2,3,4,5).
The maximum value is obtained on the boundary $x_1=8$.
X_1=[8.00000 00000 00000 0, 8.00000 00000 00000 0],
X_2=[-9.35828 66332 95584 2, -9.35828 66332 94234 0],
X_3=[-4.57207 78818 50185 3, -4.57207 78818 17701 7],
X_4=[3.59212 96114 94123 9, 3.59212 96115 93839 5],
X_5=[-2.84008 63924 86292 4, -2.84008 63924 81793 2],
F(max)=[24139.85650 22284 47 , 24139.85650 22284 95].

Case (iii): $-10 \leq x_1 \leq 8$, $-10 \leq x_i \leq 12$, (i=2,3,4), $-10 \leq x_5 \leq 10$.
The maximum value is obtained on the boundary $x_2=x_3=x_4=12$.
X_1=[-7.42310 73996 74397 9, -7.42310 73996 74397 9],
X_2=[12.00000 00000 00000 , 12.00000 00000 00000],
X_3=[12.00000 00000 00000 , 12.00000 00000 00000],
X_4=[12.00000 00000 00000 , 12.00000 00000 00000],
X_5=[-2.84008 63924 84044 7, -2.84008 63924 84044 7],
F(max)=[90193.85088 59564 90 , 90193.850088 59567 52].

Example 4: Equality constrained problem.
Maximize

$$(19) \qquad f(x_1,x_2,x_3)=-(x_1+x_2-x_3-1)^2-(x_1+x_2)^2-5x_1^2$$

subject to $2x_1+x_3=0$, $-10 \leq x_i \leq 10$, (i=1,2,3).
The computed result is:
X_1=[0.14285 71428 57138 21, 0.14285 71428 57148 04],
X_2=[0.21428 57142 85700 95, 0.21428 57142 85725 15],
X_3=[-0.28571 42857 14316 56, -0.28571 42857 14258 47],
F(max)=[-0.35714 28571 42908 72, -0.35714 28571 42803 86].
The Lagrange-multiplier obtained is:
P =[-0.71428 57142 85766 82, -0.71428 57142 85662 45].

Example 5: Inequality constrained problem.
Maximize

(20) $f(x_1,x_2)=2x_1{}^2-2x_1x_2+2x_2{}^6-6x_1$

subject to $3x_1+4x_2 \leqq 6$, $-10 \leqq x_i \leqq 10$, $(i=1,2)$.
The computed result is:

X_1 =[1.45945 94594 59459 4 , 1.45945 94594 59459 4],
X_2 =[0.40540 54054 05405 44, 0.40540 54054 05405 46],
F(max)=[5.35135 13513 51348 2 , 5.35135 13513 51354 0],

The values of Lagrange-multiplier p and slack valiable x_3 are:

X_3 =[0.0 , 0.10297 26133 56609 95D-24],
P =[0.32432 43243 24324 40, 0.32432 43243 24324 40].

V. CONCLUSION

We described an algorithm for maximizing functions by use of interval analysis. It enables us to obtain the maximum in the domain or on the boundary. Both unconstrained and constrained global maximum can be computed. So far we have calculated maxima of the functions up to five variables. If effective devices for reducing interval width of functions are developed, this method can be applied to higher-dimensional problems.

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