ON AN INTERVAL COMPUTATIONAL METHOD FOR FINDING THE REACHABLE SET IN TIME-OPTIMAL CONTROL PROBLEMS

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1. INTRODUCTION

Consider the time-optimal control of a system described by the differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.1}$$

with fixed initial data $x(t_o) = x_o$, where x is the n-dimensional state vector, u is the r-dimensional vector control function, $A(t)$, $B(t)$ are the $n\times n$, $n\times r$ - matrix functions, respectively. $A(t)$, $B(t)$, are assumed to be piecewise continuous for $t \geqq t_o$ on any finite interval.

A control function $u(t)$ is said to be admissible if it is measurable over any finite interval and takes its values from a given compact set U of E^r.

Let U denote the set of all admissible control functions. Given an initial state

$$x(t_o) = x_o \tag{1.2}$$

and a control function $u(t) \in U$, $t_o \leqq t \leqq t_1$, then equation (1.1) has a unique solution $x(t,u)$.

Let $\Phi(t)$ be the principle matrix solution of the homogeneous system

$$\dot{x}(t) = A(t)x(t) \tag{1.3}$$

satisfying $\Phi(t_o) = J$ - the identity matrix. Then, for an admissible control u, the solution of (1.1) is given by

$$x(t,u) = \Phi(t)\Phi^{-1}(t_o)x_o + \Phi(t)\int_{t_o}^{t} \Phi^{-1}(\tau)B(\tau)u(\tau)d\tau. \tag{1.4}$$

Applying the change of coordinates in equation (1.1), defined by the transformation $x = \Phi(t)y$, we get that system (1.1) is equivalent to

(1.5) $\dot{y} = \psi(t)u(t)$

with the initial data $y(t_o,u) = 0$, where $\psi(t) = \Phi^{-1}(t)B(t)$.

Define

(1.6) $R(t) = \{y(t,u);\ u \text{ measurable},\ u(\tau) \in U \text{ for } \tau \in [t_o,t]\}$

where

$$y(t,u) = \int_{t_o}^{t} \psi(\tau)u(\tau)d\tau.$$

$R(t)$ is called a reachable set.

The time-optimal control problem is to find an admissible control u^*, subject to its constraints, in such a way that the solution $x(t,u^*)$ of (1.1) reaches a continuously moving target in E^n in minimum time $t^* \geq t_o$.

The equivalent statement of the time-optimal control problem is to find an admissible control u for which $w(t) \in R(t)$ for a minimum value of $t \geq t_o$, where $w(t)$ stands for a moving target at time t.

2. PROPERTIES OF THE REACHABLE SET

We are now restricting ourselves to values of the control function in the unit cube C^r of E^r. The set of admissible controls on $[t_o,t]$ is given by

(2.1) $\Omega[t_o,t] = \{u\colon u \text{ measurable on } [t_o,t],\ u(\tau) \in C^r,\ t_o \leq \tau \leq t\}.$

The reachable set $R(t)$ is then of the form

(2.2) $R(t) = \{\int_{t_o}^{t} \psi(\tau)u(\tau)d\tau;\ u \in \Omega[t_o,t]\}.$

It is known [2] that if Ψ is an $n\times r$-matrix-valued function with components ψ_{ij} in $L_1[t_o,t^*]$ and Ω is the set of r-vector--valued measurable functions u whose components u_j satisfy $|u_j(t)| \leq 1$, $j = 1,2,\ldots,r$, and Ω^o is that subset of Ω for which $|u_j(t)| = 1$, then

$$\{\int_{t_o}^{t^*} \Psi(\tau)u(\tau)d\tau,\ u \in \Omega\} \text{ is convex, compact, and}$$

$$(2.3) \qquad \{\int_{t_o}^{t} \psi(\tau)u(\tau)d\tau; \quad u \in \Omega\} = \{\int_{t_o}^{t} \psi(\tau)u^o(\tau)d\tau; \quad u^o \in \Omega^o\}.$$

From this statement we have

THEOREM 1. The reachable set $R(t)$ is convex and compact.

Formula (2.3) contains the statement which is called in control theory the "bang-bang" principle. The set of bang-bang controls on $[t_o,t]$ is

$$(2.4) \qquad \Omega^o[t_o,t] = \{u; \ u \text{ measurable}, \ |u_j(\tau)| = 1, \ j = 1,\dots,r, \ \tau \in [t_o,t]\}.$$

These are the controls which at all times utilize all the controls available. Then

$$(2.5) \qquad R^o(t) = \{\int_{t_o}^{t} \psi(\tau)u^o(\tau)d\tau, \ u^o \in \Omega^o[t_o,t]\}$$

is the set of points reachable by the bang-bang control.

THEOREM The Bang-Bang Principle :

$$(2.6) \qquad R(t) = R^o(t) \quad \text{for each} \quad t \geq t_o.$$

The bang-bang principle says that any point that can be reached by an admissible control in time t can also be reached by a bang-bang control in the same time

There are several other properties of the reachable set.

THEOREM 2. $R(t)$ is a continuous function on $[t_o,\infty)$.

P r o o f. Since, for each $t_o \geq 0$ and $t \geq t_o$, we have

$$|y(t,u) - y(t_o,u)| = |\int_{t_o}^{t} \psi(\tau)u(\tau)d\tau| \leq |\int_{t_o}^{t} \|\psi(\tau)\|d\tau|,$$

therefore, by the definitions of the reachable set and of the metric space,

$$\rho(R(t),R(t_o)) \leq |\int_{t_o}^{t} \|\psi(\tau)\|d\tau.$$

Since $\int_{t_o}^{t} \|\psi(\tau)\|d\tau$ is absolutely continuous, the theorem is true.

THEOREM 3. If y is an interior point of $R(t^*)$ for some $t^*>t_o$, then it is an interior point of $R(t)$ for some $t \in (t_o,t^*)$.

P r o o f. Let V be a neighbourhood of y of radius δ inside $R(t^*)$. Suppose, for each $t \in (t_o,t^*)$, that y is not an interior point of $R(t)$. Then there is a support hyperplane $p(\eta)$ through y, such that $R(t)$ lies on one side of $p(\eta)$. Let the neighbourhood V of y be inside $R(t^*)$; then there is a point q of $R(t^*)$ whose distance from $R(t)$ is at least δ for each $t \in (t_o,t^*)$. This contradicts the continuity of $R(t)$ and completes the proof.

Let a nonzero vector η define a direction in E^n. Suppose that we want to find an admissible control u that maximizes the rate of change of $y(t,u)$ in the direction η, that is, we want to maximize

$$(2.7) \qquad \eta'\dot{y} = \eta'\psi(t)u(t).$$

We see that if u^* is of the form

$$(2.8) \qquad u^*(t) = \operatorname{sgn}\,[\eta'\psi(t)], \qquad \eta \neq 0,$$

then

$$\eta'\psi(t)u^*(t) = \sum_{j=1}^{r} |[\eta'\psi(t)]_j|.$$

Equation (2.8) means that, for each $j = 1,\ldots,r$,

$$u_j^*(t) = \operatorname{sgn}\,[\eta'\psi(t)]_j \quad \text{when} \quad [\eta'\psi(t)]_j \neq 0.$$

When $|u_j^*(t)| = 1$ almost everywhere for $j = 1,\ldots,r$, we say that the control u^* is bang-bang.

Thus, the control u^* maximizes $\eta'y(t,u)$ over all admissible controls if and only if it is of form (2.8). So, for any fixed $t^* >$ $> t_o$ and any u^* of form (2.8), the point $q^* = y(t^*,u^*)$ lies on the boundary of $R(t^*)$. Moreover, $\eta'(s - q^*) \leq 0$ for all $s \in R(t^*)$ and the hyperplane $p(\eta)$ through q^* normal to η is a support plane of $R(t^*)$ at q^*. Note also that if u is any other control of form (2.8), then $y(t^*,u)$ lies on this hyperplane $p(\eta)$. Conversely, if q^* lies on the boundary of $R(t^*)$, then there is a support plane $p(\eta)$ of $R(t^*)$ through q^* and we may take η, which is a nonzero vector, being an outward normal. Hence we have proved the following

THEOREM 4. A point $q^* = y(t^*,u^*)$ is a boundary point of $R(t^*)$ with η an outward normal to a support plane of $R(t^*)$ through q^* if and only if u^* is of the form $u^*(t) = \operatorname{sgn}\,[\eta'\psi(t)]$ on $[t_o,t^*]$

for some $\eta \neq 0$.

Define

$$E_j(\eta) = \{t:\ \eta' y_j(t) = 0, \quad t \in [t_o, t^*]\}.$$

We say that system (1.1) is normal on $[t_o, t^*]$ if $E_j(\eta)$ has measure zero for each $j = 1,\dots,r$ and each $\eta \neq 0$.

It is known [2] that system (1.1) is normal on $[t_o, t^*]$ if and only if $R(t^*)$ is strictly convex.

3. A COMPUTATIONAL METHOD

As it follows from the considerations above, in the computational realization we have to compute the minimum time $t^* = \inf\{t,\ p \in R(t)\}$ and to find the reachable set $R(t)$ at time t^*.

We now want to present an algorithm for finding the minimum time t^*.

Let $\alpha(\eta, t)$ be a function defined by

$$\alpha(\eta, t) = \eta'\beta(t) - \eta' w(t) \quad \text{for} \quad t \geq t_o, \qquad \eta \in E^n, \tag{3.1}$$

where

$$\eta'\beta(t) = \max_{y \in R(t)} \eta' y.$$

It can easily be seen that a necessary and sufficient condition for $w(t')$ to belong to $R(t')$ for some t' is

$$\alpha(\eta, t') \geq 0 \quad \text{for each} \quad \eta \in E^n.$$

The function $\alpha(\eta, t)$ is continuous, hence the set

$$T = \{t:\ \alpha(\eta, t) < 0\} \tag{3.2}$$

is open for any $\eta \in E^n$ and can be composed as follows:

$$T = \bigcup_{n=1}^{\infty} (a_n, b_n). \tag{3.3}$$

Let $\tau \geq t_o$, and let N_τ denote the set of indices m for which

$$a_m < \tau, \qquad b_m > t_o \quad \text{if} \quad \tau > t_o,$$

and

$$a_m \leq \tau, \qquad b_m > t_o \quad \text{if} \quad \tau = t_o.$$

Let $p(\eta,\tau)$ be defined for $\eta \in S$ (S - the unit sphere of E^n) and $\tau \geq t_o$ as

$$(3.4) \qquad p(\eta,\tau) = \begin{cases} \sup\limits_{m\in N} b_m & \text{if } N_\tau \neq \emptyset \\ \\ \tau - \alpha(\eta,\tau) & \text{if } N_\tau = \emptyset. \end{cases}$$

If it is known that $w(t) \notin R(t)$, $t_o \leq t < \tau_o$ and, for some $\eta \in S$, $\tau_1 = p(\eta,\tau_o) > \tau_o$, then $w(t) \notin R(t)$, $t_o \leq t < \tau_1$.

Let (η_k) $(k = 0,1,\ldots,\eta_k \in S)$ and t_o be given.

Assign the sequence (τ_k) to them as

$$\begin{array}{ll} \tau_o = t_o, & t_1 = p(\eta_o,\tau_o), \\ \tau_1 = \max\{t_1,\tau_o\}, & t_2 = p(\eta_1,\tau_1), \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \tau_n = \max\{t_n,\tau_{n-1}\}, & t_{n+1} = p(\eta_n,\tau_n). \end{array}$$

The sequence (τ_k) is nondecreasing and bounded with respect to (η_i) as a consequence of the assumption that there exists a solution of problem (1.1). Therefore $\lim\limits_{i\to\infty} \tau_k$ exists.

DEFINITION 1. The sequence (η_k) is said to be maximizing if, for the corresponding sequence (τ_i^*), the equality

$$(3.5) \qquad \tau^* = \lim_{k\to\infty} \tau_k^* = \sup_{(\eta_j)\in S} (\lim_{k\to\infty} \tau_k) = \tau^o$$

holds.

It is easily proved that there exists a maximizing sequence.

THEOREM 5. Let τ^* be determined by a maximizing sequence. Then $\tau^* = t^*$ is the optimal time of the control problem.

P r o o f. Suppose the contrary. Then only the inequality $\tau^* < t^*$ can hold, hence $w(t) \notin R(t)$ for $t \in [t_o,\tau^*]$. Thus there exists an η^o for which $-\varepsilon = \alpha(\eta^o,\tau^*) < 0$ with some positive ε. Since $\alpha(\eta,t)$ is continuous, there exists a neighbourhood V of (η^o,τ^*) such that, for any $(\eta,\tau) \in V$, we have

$$|\alpha(\eta,\tau) - \alpha(\eta^o,\tau^*)| < \frac{\varepsilon}{2}$$

and

$$\alpha(\eta,\tau) < -\frac{\varepsilon}{2}.$$

Moreover, there exists an n_o such that $\tau_n^* \in (\tau',\tau")$ for $n \geqq n_o$. Consider any $\bar{\eta} \in V_{\eta_o}$ and let

$$\tilde{\eta}_k = \begin{cases} \eta_k^*, & k < n_o \\ \bar{\eta}, & k \geqq n_o. \end{cases}$$

From $p(\bar{\eta},\tau_{n_o}) > \tau" > \tau^* = \tau^o$ it follows that $\lim_{k\to\infty} \tilde{\tau}_k > \tau^o$ for $(\tilde{\eta}_k)$, which contradicts the definition of τ^o.

It is very difficult to compute the set $R(t)$. For the purpose, using formula (2.2), we have to know the optimal control u^* with all its switching points. In the general case, it is not possible to find exactly all the switching points. In order to avoid these difficulties, we shall use the interval integrals.

Let F denote a function on $[0,\infty)$ to the vector interval space. Using the definition of the interval integral, we define the vector interval integral.

DEFINITION 2. If $F = (f_1,\ldots,f_n)$ is a vector interval function defined on the interval $[a,b]$, then the vector integral of F over $[a,b]$ is defined to be the interval vector

$$(3.6) \qquad \int_a^b F(t)dt = \left(\int_a^b f_1(t)dt,\ldots,\int_a^b f_n(t)dt\right)$$

where $\int_a^b f_i(t)dt$, $i = 1,\ldots,n$, denote the interval integrals.

Define

$$(3.7) \qquad [\varphi_{ij}(t)]_{n\times r} = \Phi^{-1}(t)B(t).$$

THEOREM 6. Let us assume that system (1.1) is normal; then, for any nonzero vector η, there exists an optimal control u^* of the form

$$(3.8) \qquad u^*(t) = \text{sgn}\,[\eta'\Phi^{-1}(t)B(t)]$$

for which the reachable set at time t^* satisfies the best possible inclusion

$$(3.9)\qquad R(t^*) \subset \left([-\int_{t_o}^{t^*} f_1(\tau)d\tau, \int_{t_o}^{t^*} f_1(\tau)d\tau], \ldots, [-\int_{t_o}^{t^*} f_n(\tau)d\tau, \int_{t_o}^{t^*} f_n(\tau)d\tau]\right)$$

where $\Phi(t)$ is the principle matrix solution of homogeneous system (1.3), $\Phi^{-1}(t)B(t)Q = [\varphi_{ij}(t)]_{n\times r}Q$ stands for an extended vector interval function, Q is the r-dimensional interval vector whose coordinates are the intervals of the form $[-1,1]$,

$$f_i(t) = \varphi_{i1}(t) + \ldots + \varphi_{ir}(t), \qquad i = 1,2,\ldots,n.$$

P r o o f. The result follows from a series of simple observations. Since system (1.1) is normal, the boundary points of $R(t^*)$ can, and can only, be reached by a control that is bang-bang. By theorem 4, there exists an optimal control of form (3.9). Thus, the optimal control is the r-dimensional vector function whose coordinates take the values -1 or 1. Consequently, the optimal control will be contained in the interval vector Q whose coordinates are the intervals of the form $[-1,1]$. Multiplying the matrix $\Phi^{-1}(t)B(t)$ by the interval vector Q, we get the vector interval function. Applying the methods for integration of interval functions and the Bang-Bang Principle, we obtain formula (3.9). On the other hand, since system (1.1) is normal, it follows that the reachable set $R(t^*)$ is strictly convex, so the best possible inclusion (3.9) holds.

EXAMPLE. Consider a simple control system $\dot{x}_1 = u_1$, $\dot{x}_2 = u_2$, $|u_1| \le 1$, $|u_2| \le 1$. Here

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The vector interval function will be of the form

$$\Phi^{-1}(t)B(t)Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix}.$$

The reachable set $R(t^*)$ is given by the formula

$$R(t^*) = \begin{pmatrix} [-t^*, t^*] \\ [-t^*, t^*] \end{pmatrix}.$$

This means that $R(t^*)$ is a square with sides of length $2t^*$.

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