INTERVAL OPERATORS AND FIXED INTERVALS

by R. Krawczyk

1. Introduction

In order to enclose a solution x^* of a nonlinear system of equations $g(x) = 0$, where $g: B \subseteq \mathbb{R}^n \to \mathbb{R}^n$, many interval operators $F: \mathbb{I}B \to \mathbb{I}\mathbb{R}^n$ with the property

$$x^* \in X \Rightarrow x^* \in F(X) \qquad (*)$$

are discussed.

By applying the iteration method

$$X_0 \subseteq B, \; X_{k+1} := F(X_k), \; k = 0,1,2,\ldots$$

we obtain a monotone sequence of intervals

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$$

We distinguish between two cases:

1. There exists a $k \in \mathbb{N}$ with $X_{k+1} = \emptyset$. Then, because of (*) X_0 contains no solution.

2. The sequence $\{X_k\}$, is infinite. It then follows that $\lim_{k \to \infty} X_k = X_\infty$.

2.1 Additionally, if $\operatorname{rad} X_\infty = 0$ then $X_\infty = x^*$ is a unique solution.

2.2 On the other hand, if a solution $x^* \in X_0$ exists then it follows that $x^* \in X_k$ for all $k \in \mathbb{N}$ and $x^* \in X_\infty$.

In all cases, we assume that the Jacobi matrix g' of g exists, and that we know an interval extension G' of g', or more generally, that g fulfills an interval Lipschitz condition

$$g(x_1) - g(x_2) \in L(X)(x_1 - x_2), \quad x_1, x_2 \in X \in \mathbb{I}B.$$

In many papers special interval operators for F are described, and questions about existence and uniqueness of a solution x^* or the question: "under which assumptions do we get $X_\infty = x^*$?" are answered.

(Some basic papers of this subject are: [3], [4], [6], [10], [12], [20], [21], [23], [24], [25], [27], [28] and [29]. See also the references of [13].)

Adams [1] and Gay [8], [9] have extended these studies to the case that g is not exactly known (e. g., if the coefficients of g are intervals). They thereby start from the function $g: B \subseteq \mathbb{R}^n \times D \subseteq \mathbb{R}^p \to \mathbb{R}^n$. If $x^*(d)$ denotes a zero of $g(x,d) = 0$ with a fixed $d \in D$, then they define a set of solutions X^* by $X^* := \{x^*(d) \mid d \in D\}$, and they give bounds or intervals, respectively, which enclose X^*.

In another model we use a function strip $G: B \subseteq \mathbb{R}^n \to \mathbb{I}\mathbb{R}^n$ instead of a function g, and instead of a zero we get a zero set X^*, which can be enclosed with the help of a fixed interval of an interval operator F, or pseudofixed interval, respectively. (See [7], [14], [15], [16], [17].)

2. Notation and basic concepts

Lower case letters denote real values (vectors, matrices and real-valued functions). Capital letters denote sets (interval vectors, interval matrices and interval functions). $\mathbb{I}\mathbb{R}^n$ [or $\mathbb{I}\mathbb{R}^{n\times n}$, respectively] denotes the set of all interval vectors [or interval matrices, respectively], and $\mathbb{I}B := \{X \in \mathbb{I}\mathbb{R}^n \mid X \subseteq B\}$.

If Σ is a bounded subset of $\mathbb{R}^n$, we denote by $\square\Sigma := [\inf \Sigma, \sup \Sigma]$ the <u>interval hull</u> of Σ.

Let $X = [\underline{x},\bar{x}] \in \mathbb{I}\mathbb{R}^n$; then $\operatorname{rad} X := \frac{1}{2}(\bar{x}-\underline{x})$ denotes the <u>radius,</u> $\operatorname{mid} X = \check{x} = \frac{1}{2}(\underline{x}+\bar{x})$ the <u>midpoint</u> and $|X| := \sup(\bar{x},-\underline{x})$ the <u>absolute value</u> of X. Analogous notations apply to $L = [\underline{l},\bar{l}] \in \mathbb{I}\mathbb{R}^{n\times n}$. If $r \in \mathbb{R}^{n\times n}$, then $\sigma(r)$ denotes the spectral radius of r. Concerning interval arithmetic we refer to [5] and [19].

By Neumaier [22] a map $S: \mathbb{I}\mathbb{R}^n \to \mathbb{I}\mathbb{R}^n$ is called <u>sublinear</u> if the following axioms are valid for all $X,Y \in \mathbb{I}\mathbb{R}^n$.

(S1)	$X \subseteq Y \Rightarrow SX \subseteq SY$		(inclusion isotonicity),
(S2)	$\alpha \in \mathbb{R} \Rightarrow S(X\alpha) = (SX)\alpha$		(homogeneity),
(S3)	$S(X+Y) \subseteq SX + SY$		(subadditivity).

We extend S to matrix arguments by applying it to each column of the matrix. Moreover, we set

$$\kappa(S) := Se \quad\text{and}\quad |S| = |SE|,$$

where e denotes the unit matrix, and $E = [-e,e]$. (In [22] the interval matrix $\kappa(S) \in \mathbb{I}\mathbb{R}^{n\times n}$ is called the kernel and the nonnegative matrix $|S|$ is called the absolute value of S).

A sublinear map is called normal, if for all $X \in \mathbb{I}\mathbb{R}^n$,

(S4) $\quad \mathrm{rad}(SX) \geq |S|\,\mathrm{rad}\,X;$

it is called centered, if

(S5) $\quad X \in \mathbb{I}\mathbb{R}^n,\ \mathrm{mid}(SX) = 0 \Rightarrow \mathrm{mid}\,X = 0,$

and regular, if

(S6) $\quad x \in \mathbb{R}^n,\ 0 \in Sx \Rightarrow x = 0.$

Let $L \in \mathbb{I}\mathbb{R}^{n\times n}$ be a regular interval matrix (i. e., each matrix $l \in L$ is regular). Then L^{-1} is defined by

$$L^{-1} := \square\{l^{-1} \mid l \in L\}.$$

Moreover, a sublinear map L^I is called inverse of L, if

$$l^{-1}x \in L^I X \quad\text{for all}\quad l \in L,\ x \in X.$$

$L := \{L_{ik}\} \in \mathbb{I}\mathbb{R}^{n\times n}$ is called H-matrix, if the real matrix $\langle L\rangle := \{l_{ik}\}$ with $l_{ii} := \inf\{|l| \mid l \in L_{ii}\}$ and $l_{ik} := -|L_{ik}|$ for $i \neq k$, $i,k = 1(1)n$, is an M-matrix.

3. A function strip and its zero set

Let $G\colon B \subseteq \mathbb{R}^n \to \mathbb{I}\mathbb{R}^n$ be a map which associates with each $x \in B$ an interval

$$G(x) := [\underline{g}(x), \overline{g}(x)]. \tag{1}$$

Such a map is called a function strip. We call

$$X^* := \{x \in B \mid \underline{g}(x) \leq 0 \leq \overline{g}(x)\}$$

the zero set of G (which can be empty).

Remark: This zero set X^* encloses the set of solutions defined by Adams [1] or Gay [9], respectively.

We assume that G on each $X \in \mathbb{I}B$ satisfies an interval Lipschitz condition, i. e., the real functions $\underline{g}$ and $\bar{g}$ both satisfy the same interval Lipschitz condition

$$\left.\begin{array}{l} \underline{g}(x_1) - \underline{g}(x_2) \in L(X)(x_1-x_2), \quad \bar{g}(x_1) - \bar{g}(x_2) \in L(X)(x_1-x_2) \\ \quad \text{for all } x_1, x_2 \in X \in \mathbb{I}B. \end{array}\right\} \qquad (2)$$

We call $L: \mathbb{I}B \to \mathbb{I}\mathbb{R}^{n\times n}$ a Lipschitz operator and assume that L is inclusion isotone, i. e.

$$X \subseteq Y \Rightarrow L(X) \subseteq L(Y). \qquad (3)$$

4. Interval operators of a function strip, properties of such operators and some general theorems

Let the map $F: \mathbb{I}B \to \mathbb{I}\mathbb{R}^n$ be a continuous interval operator. We call $\hat{X} \in \mathbb{I}B$ a fixed interval of F if $F(\hat{X}) = \hat{X}$, and we call $X \in \mathbb{I}B$ a pseudo-fixed interval of F, if $F(X) \supseteq X$.

Properties of an interval operator (see definition in [15]):
Let $X, Y \in \mathbb{I}B$, $X^* \subseteq B$ the zero set of a function strip G and $\hat{X} \in \mathbb{I}B$ a fixed interval of an interval operator F. Then we call F

(E1)	inclusion isotone,	if	$X \subseteq Y \Rightarrow F(X) \subseteq F(Y)$,
(E2)	normal,	if	$X^* \subseteq \hat{X}$,
(E3)	inclusion preserving,	if	$X^* \subseteq X \Rightarrow X^* \subseteq F(X)$,
(E4)	fixed interval preserving,	if	$\hat{X} \subseteq X \Rightarrow \hat{X} \subseteq F(X)$,
(E5)	strong,	if	$F(X) \supseteq X \supseteq \hat{X} \Rightarrow X = \hat{X}$.

The following theorems are valid.

Theorem 1: If the continuous interval operator F is inclusion preserving and $\emptyset \neq X^* \subseteq X_0 \subseteq B$ [or fixed interval preserving and $\hat{X} \subseteq X_0 \subseteq B$, respectively], then the interval sequence $\{X_k\}$ defined by

$$X_{k+1} := X_k \cap F(X_k), \quad k = 0, 1, 2, \ldots \qquad (4)$$

converges; hence

$$\left.\begin{array}{l}\lim_{k\to\infty} X_k = X_\infty \supseteq X^* \quad [\text{or } X_\infty \supseteq \hat{X}, \text{ respectively}] \\ \qquad \text{with} \quad F(X_\infty) \supseteq X_\infty\end{array}\right\} \tag{5}$$

holds.

(See Theorem 2.3 in [15]).

Theorem 2: If the continuous interval operator F is fixed interval preserving and strong, and if F possesses a fixed interval $\hat{X} \subseteq X_0$, then we get for the interval sequence $\{X_k\}$ defined by (4)

$$\lim_{k\to\infty} X_k = \hat{X}.$$

(See Theorem 2.4 in [15]).

Theorem 3: If the continuous interval operator F is inclusion isotone, and if $F(X) \subseteq X$ then there exists a fixed interval $\hat{X}$ of F.

In the following sections we discuss three classes of special interval operators.

5. Newton-like interval operators

$$N_0(X) := \check{x} - L^I G(\check{x}), \tag{6}$$

where $L := L(X_0)$ is a constant Lipschitz matrix of (2), and L^I denotes a normal and centered inverse of L.

Theorem 4 (conclusion from Theorem 4.2 in [17]): Let N_0 be defined by (6), then N_0 is normal and inclusion preserving.

Supposing, $\hat{X} := [\hat{x} - \text{rad}\,\hat{X},\ \hat{x} + \text{rad}\,\hat{X}]$ is a fixed interval of F. Then it follows from (S5) that

$$\text{mid}\, G(\hat{x}) = 0, \tag{7}$$

and from (S4), as well as (3.8) in [17] that

$$\text{rad}\, X = |L^I|\, \text{rad}\, G(\hat{x}). \tag{8}$$

This means: the absolute value (matrix) $|L^I|$ determines the "size" of a fixed interval.

A second "measure" is the matrix

$$\mathrm{rad}(L^I L),$$

which we call <u>convergence matrix,</u> because it is responsible for the speed of convergence of the iteration (4). Moreover, the following statement holds:

<u>Theorem 5:</u> Let N_0 be defined by (4). If a fixed interval $\hat{X}$ of N_0 exists, and if

$$\sigma(\mathrm{rad}(L^I L)) < 1, \tag{9}$$

then N_0 is a strong operator, i. e., the property (E5) is satisfied. (See Proposition 6.2 in [17]).

Next we discuss four examples of an inverse L^I of L.

1. $L^G Z := \mathrm{IGA}(L,Z)$,

   where IGA denotes the interval Gauss algorithm (see [6]). Sufficient conditions for the existence of L^G are:

   (i) L = regular and n = 2

(see Reichmann [30]).

   (ii) L = H-matrix

(see Alefeld [2]).

Generally, the regularity of L is not sufficient for the existence of L^G. (See a variant of Reichmann [30] in the remark 3 of Theorem 3 in [22]).

For the following three inverses L^I we assume that the Lipschitz-matrix $L \in \mathbb{IR}^{n\times n}$ is <u>strongly regular</u>, i. e. by (7.1) in [17]:

The matrix

$$a := (\mathrm{mid}\, L)^{-1} \tag{10}$$

exists, and with

$$r := |a|\,\mathrm{rad}\, L \tag{11}$$

the condition

$$\sigma(r) < 1 \tag{12}$$

holds. Then

$$q := (e-r)^{-1}r \tag{13}$$

exists, and it is a nonnegative matrix.

2. $L^I = L^K$: $L^K Z := aZ + (qE)(aZ)$, (14)

3. $L^I = L^V$: $L^V Z := [e-q,\ e+q](aZ)$, (15)

4. $L^I = L^P$: $L^P Z := (aL)^G (aZ)$. (16)

Remark: For the one-dimensional case, and if G degenerates to a function g, L^G was introduced by Moore [19] and applied by Nickel [27] and many other authors. Among other things, the multi-dimensional case was discussed by Alefeld/Herzberger [4]. New results for L^G were derived by Neumaier [22] and [26]. For the function strip, L^K was introduced by Krawczyk [14], and L^V by Krawczyk/Neumaier [16]. L^P means preconditioning of L with $(\text{mid}\,L)^{-1}$ (see section 6 in [22]), which was applied by Hansen/Smith [11].

Theorem 6: If $\sigma(q) < 1$, where q is defined by (13), then the inverses L^K, L^V and L^P are regular.
(See examples 2, 3 and 4 of section 7 in [17]).

Theorem 7: Let N_0 be defined by (6) with $L^I = L^K$ (see (14)). Then N_0 is a fixed interval preserving operator.
(See Theorem 5.4 in [15]).

Remark: It is not necessary, however, that N_0 with L^G or L^V, L^P, respectively be fixed interval preserving, as the following example shows:

Let be $\underline{g}(x) := \begin{cases} 4x-6, & \text{if } x \geq 0, \\ 2x-6, & \text{if } x < 0, \end{cases}$ $\quad \bar{g}(x) = 4x+6.$
Then $L = [2, 4]$ and $L^G Z = L^P Z = Z \times \left[\frac{1}{4}, \frac{1}{2}\right]$. $\hat{X} = [-3, 3]$ is a fixed interval of N_0 with $L^I = L^G$. Choosing $X_0 = [-3, 5] \supseteq \hat{X}$ we obtain $X_1 = [-3, 2] \not\supseteq \hat{X}$, which is contrary to the statement of Theorem 7.

(As far as L^V is concerned, see example 5.3 in [15]).

Theorem 8: Let N_0 be defined by (6) with $L^I = L^K$, and let $\sigma(q) < 1$, where q is defined by (13). Then N_0 is a strong operator.
(See Theorem 5.5 in [15]).

Remark: In comparing this result with Theorem 5 we can say: $\sigma(q) < 1$ is a weaker assumption than (9) that is, $\sigma(\mathrm{rad}(L^I L)) < 1$, because $\mathrm{rad}(L^K L) = 2q$.

Comparison of the cases 1., 2. and 3.:

$$\text{(i)} \quad L^V Z \subseteq L^K Z, \quad L^P Z \subseteq L^K Z \quad \text{for all} \quad Z \in \mathbb{I}\mathbb{R}^n , \tag{17}$$

$$\text{(ii)} \quad |L^K| = |L^V| = |L^P| = (e-r)^{-1}|a| , \tag{18}$$

$$\text{(iii)} \quad \mathrm{rad}(L^K L) = 2q , \tag{19}$$

$$\text{(iv)} \quad q \leq \mathrm{rad}(L^V L) \leq 2q , \tag{20}$$

$$\text{(v)} \quad q \leq \mathrm{rad}(L^P L) \leq 2q . \tag{21}$$

From (17) it follows that the application of L^V and L^P yields better results than the use of L^K. However, we cannot tell whether L^V or L^P is more favorable. A comparison with L^G is difficult, because the fixed interval of N_0^G (applying L^G in (6)) generally does not coincide with the fixed interval N_0^K (applying L^K in (6)).

However, it follows from (18) that all interval operators: N_0^K, N_0^V and N_0^P possess the same fixed interval $\hat{X}$.

Conclusion from Theorem 8: If the assumptions of Theorem 8 are fulfilled then N_0 with $L^I = L^V$ or $L^I = L^P$, respectively, is strong. Because of (17) it follows that $N_0^V(X) \subseteq N_0^K(X)$, as well as $N_0^P(X) \subseteq N_0^K(X)$. ($N_0^V$ denotes the operator (6) with $L^I = L^V$, and N_0^P is the notation if $L^I = L^P$. If a fixed interval $\hat{X}$ of N_0^K exists, then by (18) $\hat{X}$ is a fixed interval of N_0^V and N_0^P, too. Therefore $N_0^V(X) \supseteq X \supseteq \hat{X}$ implies $N_0^K(X) \supseteq X \supseteq \hat{X}$, and by applying Theorem 7 we obtain $X = \hat{X}$. Analogously, $N_0^P(X) \supseteq X \supseteq \hat{X}$ implies $X = \hat{X}$.

Remark: Theorem 8 and the conclusions are true only if L^I is constant. However, it is not necessary that the interval operator

$$N(X) := \check{x} - L^I(X)G(\check{x}) \tag{22}$$

with variable $L(X)$ is strong, as the examples 5.4 and 5.5 in [15] show. Furthermore, there can exist more than one fixed interval which all have the same midpoint $\hat{x}$. In contrast, N_0 has at most one fixed interval if $\sigma(r) < 1$, since the zero $\hat{x}$ of the equation $\mathrm{mid}\, G(x) = 0$ is unique, and by (8), $\mathrm{rad}\, \hat{X}$ is independent of X (see example 6.1 in [16]). It is even possible that there exists a fixed interval of N but not of

N_0 (see example 6.3 in [16]). The contrary statement is not true. If N_0 possesses a fixed interval then there exists at least one fixed interval of N (see Theorem 6.5 in [16]). If $\hat{X}_0$ denotes a fixed interval of N_0 and $\hat{X}$ a fixed interval of N, then $\hat{X} \subseteq \hat{X}_0$ holds for each fixed interval $\hat{X}$ of N (see Theorem 6.4 in [16]).

Overestimation: Let $X^* \neq \emptyset$, then the iteration method (4) with the operator (6) yields a limit interval $X_\infty \supseteq X^*$ (Theorem 1) or $\hat{X} \supseteq X^*$ (Theorem 2), respectively. The "distance" of the interval hull of X^* from $\hat{X}$ can be bounded by the following Theorem.

Theorem 9: Let L^I be a regular and centered inverse of L. Suppose that for each $l \in L$ the inequality

$$|L^I| \leqq |l^{-1}| + 2\,\mathrm{rad}\,(\kappa(L^I)) \tag{23}$$

holds. Then it follows that

$$0 \leqq \mathrm{rad}\,\hat{X} - \mathrm{rad}\,\square X^* \leqq 2(\mathrm{rad}(L^{-1}) + \mathrm{rad}(\kappa(L^I)))\,\mathrm{rad}\,G(\hat{x}) \tag{24}$$

(see Theorem (5.1), (iv) in [17]).

Remarks: 1. The assumption (23) is valid for $L^I = L^K, L^V, L^P$.

2. Since $L^{-1} \subseteq \kappa(L^I)$, the bound (24) can be simplified by

$$\mathrm{rad}\,\hat{X} - \mathrm{rad}\,\square X^* \leqq 4\,\mathrm{rad}\,(\kappa(L^I))\,\mathrm{rad}\,G(\hat{x}).$$

3. If $\mathrm{rad}(\kappa(L^I)) = O(\varepsilon)$ and $\mathrm{rad}G(\hat{x}) = O(\varepsilon)$ then it follows from (24) that $\mathrm{rad}\,\hat{X} - \mathrm{rad}\,\square X^* = O(\varepsilon^2)$. This means quadratic vonvergence, if $\varepsilon \to 0$.

6. K-operators

Instead of the Newton-like interval oprator (6) for iteration (4) we can use the operator

$$K_0(X) := \check{x} - aG(\check{x}) + (rE)(X-\check{x}), \tag{25}$$

where a and r are defined by (10) and (11).

If we assume (12) - such that $\sigma(r) < 1$ - then there exists at most one fixed interval $\hat{X}$ of K_0. By setting $\hat{X} = [\hat{x} - \mathrm{rad}\,\hat{X},\ \hat{x} + \mathrm{rad}\,\hat{X}]$ we obtain

$$\mathrm{mid}\,G(\hat{x}) = 0,\quad \mathrm{rad}\,\hat{X} = (e-r)^{-1}|a|\,\mathrm{rad}\,G(\hat{x}). \tag{26}$$

From (8) and (18) it follows that a fixed interval of N_0 with $L^I = L^K, L^V, L^P$ coincides with a fixed interval of K_0. With respect to the properties of K_0 the following theorem holds.

Theorem 10: Under the assumption (12), the interval operator K_0 defined by (25) is inclusion isotone, normal, inclusion preserving, fixed interval preserving and strong.
(See Theorem 5.1 - 5.5 in [15].)

Remarks: 1. For the statement: "K_0 is a strong operator" the assumption $\sigma(q) < 1$ is not necessary. In contrast, $\sigma(q) < 1$ is necessary for N_0 to be a strong operator.
(See example 5.4 in [15].)

2. The remark referring to the property: "strong" and to fixed intervals of (22) with $L^I(X) = L^K(X)$, $L^V(X)$, $L^P(X)$ yields an analogous result for the interval operator

$$K(X) := \check{x} - a(X)G(\check{x}) + (r(X)E)(X-\check{x}) \tag{27}$$

with $a(x) := (\text{mid } L(X))^{-1}$ and $r(X) := |a(X)| \text{ rad } L(X)$.
Each fixed interval of N(X) is a fixed interval of K(X), too, and vice versa.

In correspondence with the bound (24) with regard to the distance of the solution set X^* from a fixed interval $\hat{X}$, the inequality

$$0 \leq \text{rad } \hat{X} - \text{rad } X^* \leq (2 \text{ rad } (L^{-1}) + q|a|) \text{ rad } G(\hat{x}) \tag{28}$$

holds.

By comparing the bound (28) with (24) we obtain from (24) in the case $L^I = L^V$, because of $\kappa(L^I) = [a - q|a|, a + q|a|]$ (see example 3, (iv) in [17]),

$$\text{rad } \hat{X} - \text{rad } X^* \leq (2 \text{ rad } L^{-1} + 2q|a|) \text{ rad } G(\hat{x}),$$

which is less favorable then (28).

7. The optimal operator

Under special assumptions we can apply an operator $\mathcal{O}_0$ or $\mathcal{O}$ instead of N_0 or N, respectively, K_0 or K, which optimally encloses a generalized zero set.

Assumption: Let a matrix $b \in \mathbb{R}^{n\times n}$ exist such that

$$0 \leq e - bL(X) \quad \text{for all} \quad X \in \mathbb{I}B \tag{29}$$

holds.

b can be split as $b = b^+ - b^-$, where $b^+ := \sup(b,0)$, $b^- := \sup(-b,0)$. Let

$$\underline{f}(x) := x - b^+\overline{g}(x) + b^-\underline{g}(x),$$
$$\overline{f}(x) := x - b^+\underline{g}(x) + b^-\overline{g}(x).$$

Then the optimal operator is given by

$$\mathcal{O}_0(X) := [\underline{f}(\underline{x}), \overline{f}(\overline{x})]. \tag{30}$$

Remark: If b = constant then $\mathcal{O}_0$ is indenpendent of L(X).

Theorem 11: If the assumption (29) holds, then the operator $\mathcal{O}_0$ defined by (30) is inclusion isotone, normal, inclusion preserving and fixed interval preserving. If, in addition,

$$\sigma|e - bL(X)| < 1 \quad \text{for all} \quad X \in \mathbb{I}B, \tag{31}$$

then $\mathcal{O}_0$ is a strong operator.
(The proof of this theorem will be published later).

We call the set

$$X^{**} := \{x \in D \mid \underline{f}(x) \leq x \leq \overline{f}(x)\}$$

a pseudo-zero set; note that

$$X^* \subseteq X^{**}$$

holds.

Theorem 12: Let the assumptions (29) and (31) be fulfilled. If, additionally, $X^* \neq \emptyset$, and a fixed interval $\hat{X}$ of $\mathcal{O}_0$ exists, then the iterated sequence (4) with the operator (30) converges, and we obtain

$$\lim_{k \to \infty} X_k = \hat{X} = \square X^{**}.$$

<u>Theorem 13</u> (existence): Under the assumptions (29), (31) and $\mathcal{O}_0(X) \subseteq X$ there exists a fixed interval $\hat{X} = \square X^{**}$ of $\mathcal{O}_0$.

<u>Theorem 14</u> (existence): Under the assumptions of Theorem 13, if $X^{**} \neq \emptyset$ and $\square X^{**} \subseteq \operatorname{int} B$ *) there exists a fixed interval $\hat{X} = \square X^{**}$ of $\mathcal{O}_0$.

<u>Theorem 15</u> (overestimation): Let $X^* \neq \emptyset$ and a fixed interval of $\mathcal{O}_0$ exist, $\bar{s} := |e - bL(\hat{X})|$, $t = 2(e-\bar{s})^{-1}$ and $z := \operatorname{rad} G(\hat{x}) + \operatorname{rad} L(\hat{X}) \operatorname{rad} \hat{X}$. Then

$$\operatorname{rad} \hat{X} - \operatorname{rad} \square X^* \leq \inf\{tb^+z,\ tb^-z\} \tag{32}$$

holds.

<u>Special cases:</u> $b^- = 0$ or $b^+ = 0$: Then it follows from (32) that

$$\hat{X} = \square X^*,$$

i.e., the zero set X^* can be enclosed optimally by $\hat{X}$.

<u>Remark:</u> Let $L(X)$ be inverse nonnegative for all $X \in \mathbb{IB}$. By choosing $b = \bar{l}^{-1}(X_0) = b^+$, $b^- = 0$ we obtain the operator $\mathcal{O}_0(X) = [\underline{x} - \bar{l}^{-1}\bar{g}(\underline{x}),$ $\bar{x} - \bar{l}^{-1}\underline{g}(\bar{x})]$. If $b(X) = \bar{l}^{-1}(X)$ is variable, we then get the interval operator $\mathcal{O}(X)$ which was introduced in [7] (see (4.1) in [7]). In contrast to N_0 and N, or K_0 and K, respectively the fixed interval of $\mathcal{O}_0$ coincides with the fixed interval of $\mathcal{O}$, such that under the given assumptions there exists at most one fixed interval of $\mathcal{O}$.

If the function strip G degenerates to a real function g then we obtain the method of Li [18].

*) $\operatorname{int} B$ means the interior of B.

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