LINEAR INTERVAL EQUATIONS

A. Neumaier
Institut für Angewandte Mathematik
Universität Freiburg
Freiburg i. Br.
West Germany

Abstract. This is a short survey of theory and techniques for the solution of linear interval equations with square or rectangular coefficient matrix.

Notation. $\mathbb{IR}^n$ denotes the set of interval vectors with n components, and $\mathbb{IR}^{m\times n}$ the set of interval m×n-matrices; an interval matrix containing only one element is called thin. In this survey we shall use the terms vector and matrix as synonyms for interval vector and interval matrix. The midpoint, radius, and absolute value of a matrix $A \in \mathbb{IR}^{m\times n}$ are understood componentwise and denoted by $\check{A} = \operatorname{mid} A$, $\rho(A) = \operatorname{rad} A$, and $|A|$, respectively. Similar definitions apply for vectors. In the following, $A \in \mathbb{IR}^{m\times n}$ is a fixed interval matrix, and $b \in \mathbb{IR}^n$ a fixed interval vector.

A linear interval equation with coefficient matrix A and righthand side b is defined as the family of linear equations

$$\tilde{A}\tilde{x} = \tilde{b} \quad (\tilde{A} \in A,\ \tilde{b} \in b); \tag{1}$$

the solution set of (1) is the set

$$\Sigma(A,b) := \{\tilde{x} \in \mathbb{R}^n \mid \tilde{A}\tilde{x} = \tilde{b} \text{ for some } \tilde{A} \in A,\ \tilde{b} \in b\}.$$

By a result of Beeck [6], the solution set can also be described as

$$\Sigma(A,b) = \{\tilde{x} \in \mathbb{R}^n \mid A\tilde{x} \cap b \neq \emptyset\}. \tag{2}$$

The criterion $\tilde{x} \in \Sigma(A,b)$ iff $A\tilde{x} \cap b \neq \emptyset$ is equivalent to a famous perturbation theorem of Oettli and Prager [26].

The solution set $\Sigma(A,b)$ is bounded if A is regular, i.e. if all matrices $\tilde{A} \in A$ have rank n. A sufficient condition for the regularity of a square matrix A is (Ris [29]):

$$\check{A}^{-1} \text{ exists and } |\check{A}^{-1}|\rho(A) \text{ has spectral radius} < 1; \qquad (3)$$

let us call such matrices strongly regular. Since the solution set of a linear interval equation may be very complicated, we are interested in finding interval enclosures for $\Sigma(A,b)$. The interval vector with smallest radius containing $\Sigma(A,b)$ is the hull of the solution set,

$$A^H b := \square\Sigma(A,b) = [\inf \Sigma(A,b), \sup \Sigma(A,b)]. \qquad (4)$$

1. The square case

In this section, we treat the case of a square coefficient matrix (i.e. $m = n$).

1.1 Computing the hull. The computation of $A^H b$ is, in general, a very difficult problem; the known algorithms seem to have a worst case complexity exponential in n. However, for $n = 1$, the hull is computable by simple division, $A^H b = b/A$, and for $n = 2$, a simple method is described in Apostolatos and Kulisch [4]. For general n, several algorithms have been given by Rohn [30], [31], [32]; the algorithm of the first paper is iterative and assumes that A is strongly regular, the other two papers treat the case of general regular A. The algorithms are very time-consuming for large n; moreover, in their present form, they do not account for rounding errors in the computation.

For dimensions n larger than about 5, practical methods are available only in special cases. If A is thin then $A^H b = A^{-1} b$ (Beeck [7]), and if A is an M-matrix then Gauss-Seidel iteration yields the hull (Barth and Nuding [5]). In case that the righthand side b satisfies one of the conditions $b \geq 0$, $b \leq 0$, or $b \ni 0$, the hull can be computed for

inverse positive A as $A^H b = [\overline{A}^{-1}, \underline{A}^{-1}]b$ (Beeck [7]); and for the more special case of M-matrices, Gauss elimination gives the hull for these righthand sides (Barth and Nuding [5], Beeck [7]). Further methods for the computation of the hull for M-matrices and inverse positive matrices are described in Beeck [8] and Neumaier [24].

For other matrices we are, at present, restricted to the use of methods which do not compute an optimal enclosure.

1.2 Gauss elimination. We denote by $A^G b$ the result of Gauss elimination applied without pivoting to the linear interval equation (1). Depending on the type of the coefficient matrix, the results may be very good or very bad.

Gauss elimination is almost optimal if A is an M-matrix: We have $A^G b \subseteq [\overline{A}^{-1}, \underline{A}^{-1}]b$ (Neumaier [24]), and in special cases we get the hull (Barth and Nuding [5],Beeck [7] for $b \geqq 0$, $b \leqq 0$, or $b \ni 0$, and Schwandt [37] for thin A). But we warn that standard column pivoting may destroy the M-matrix property, leading to a loss of upto 3 decimals in accuracy (Schätzle [34])

Gauss elimination without pivoting is also reliable for diagonally dominant matrices (experiments of Kopp [17]) and Hessenberg matrices with a special sign structure (Reichmann [27]); it also works well with pivoting for n = 2 (Alefeld [3] and Reichmann [28] show that $A^G b$ exists for n = 2 iff A is regular).

Gauss elimination can also be performed without pivoting if A is an H-matrix (Alefeld [2]), and with pivoting if A is regular and $\rho(A)$ is sufficiently small (Neumaier [24]). However rounding errors and dependency may lead in these cases to catastrophic overestimation (exponential in n) or even to breakdown due to division by an interval containing zero (Wongwises [40], Schätzle [34]).

Gauss elimination can be coupled with iteration by splitting A as $A = A_0 - E$ and considering the iteration

$$x^{\ell+1} := A_0^G(Ex^\ell + b); \tag{5}$$

cf. Alefeld [3] for $A_0 = \check{A}$, Mayer [21] for more general splittings. A

fixpoint x^∞ of (5) is an enclosure for $A^H b$. If this iteration converges for all x^0, b then, by Neumaier [24], Gauss elimination cannot break down, and, if $\tilde{b} = 0$, $\operatorname{rad} A^G b \leq \operatorname{rad} x^\infty$. However, if A is an M-matrix and $A_0 = \overline{A}$ then the iteration converges for all x^0, b, and the limit is the hull, $x^\infty = A^H b$.

1.3 Gauss-Seidel iteration. If an initial enclosure x^0 for $A^H b$ is known, a nested sequence of enclosures x^ℓ for $A^H b$ can be defined by Gauss-Seidel iteration with componentwise intersection,

$$x^{\ell+1} := \Gamma(A,b,x^\ell) \quad (\ell = 0,1,2,\ldots), \tag{6}$$

where the vector $y := \Gamma(A,b,x)$ is defined by

$$\left.\begin{aligned} y_i' &:= (b_i - \sum_{k<i} A_{ik}y_k - \sum_{k>i} A_{ik}x_k)/A_{ii}, \\ y_i &:= x_i \cap y_i' \end{aligned}\right\} \ (i = 1,\ldots,n);$$

cf. Ris [29], Neumaier [23]. Clearly the method applies only when $0 \notin A_{ii}$ $(i = 1,\ldots,n)$, although it can be modified for the general case (cf. Hansen and Sengupta [14]).

If A is an M-matrix then the iteration (6) converges to the hull $A^H b$ (Barth and Nuding [5]); if A is an H-matrix it can at least be shown that the limit x^∞ is contained in a vector $A^F b$ independent of the initial enclosure x^0 (Neumaier [23]), thus guaranteeing that at least very bad enclosures will be improved. Gay [12] proved that Gauss-Seidel iteration is faster and has a smaller limit radius than the whole-step iteration

$$x^{\ell+1} = x^\ell \cap (b + (I - A)x^\ell);$$

more generally, Neumaier [23] proved the same optimality result within the class of iterations defined by triangular splittings. In particular, overrelaxation cannot improve the iteration (cf. Mayer [22], Cornelius [9]). Recently, based on ideas of Alefeld [1], a symmetric Gauss-Seidel iteration was discussed by Schwandt [35] and Shearer and Wolfe [38] in a nonlinear setting; this iteration is still faster than (6).

An easy way to get an initial enclosure for arbitrary H-matrices A uses Ostrowski's comparison matrix <A> (see Neumaier [23]) and the implication

$$u > 0,\ \langle A\rangle u \geqq v > 0 \Rightarrow A^H b \subseteq [-u,u]\,\|b\|_v\,, \tag{7}$$

where $\|b\|_v = \max\{|b_i|/v_i \mid i = 1,\ldots,n\}$. If $\beta := \|I - A\|_\infty < 1$ then one can take $v = e := (1,\ldots,1)^T$, $u = (1-\beta)^{-1}e$; in general, since $u^* := \langle A\rangle^{-1}e$ satisfies $u^* > 0$ and $\langle A\rangle u^* = e > 0$, any sufficiently good approximation u of u^* leads with $v = \langle A\rangle u$ to a valid bound.

1.4 Preconditioning. To improve the performance of Gauss elimination, Hansen and Smith [15] suggest to precondition (1) by multiplying with a matrix C (they use an approximate inverse of $\check{A}$), leading to the preconditioned system

$$\tilde{A}x = \tilde{b} \quad (\tilde{A} \in CA,\ \tilde{b} \in Cb)\,. \tag{8}$$

The preconditioning with $C = \check{A}^{-1}$ leads to a regular system (8) iff A is strongly regular (Ris [29]); in this case, CA is an H-matrix, and Gauss elimination can be performed with the matrix CA. More generally, if CA is an H-matrix for some matrix C then A must be strongly regular, and if

$$\beta := \|I - CA\| < 1 \tag{9}$$

in some scaled maximum norm then β takes its minimal value for the choice $C = \check{A}^{-1}$ (Neumaier [23]). Unless $\beta > 1$ or β is very close to 1, the overestimation inherent in the transformation from (1) to (8) is small since by Neumaier [25],

$$\|\mathrm{rad}\ (CA)^H(Cb)\| \leq \frac{1+\beta}{1-\beta}\,\|\mathrm{rad}\ A^H b\|\,. \tag{10}$$

If A is inverse positive and $C = \bar{A}^{-1}$ then we even have $(CA)^H(Cb) = A^H b$ (Neumaier [24]).

For $\beta \ll 1$, CA is almost the identity, and it is faster to compute an enclosure for the solution set of (8) by iteration. The oldest method (cf. Krawczyk [18]) uses

$$x^{\ell+1} := x^\ell \cap (Cb + (I - CA)x^\ell)\,, \tag{11}$$

where the initial enclosure x^0 is found as

$$x^0 := [-u,u] \quad (u_i = \frac{1}{1-\beta} \|Cb\|_\infty, \ i = 1,\ldots,n),$$

assuming that (9) holds in the maximum norm. Every x^ℓ is an enclosure of $A^H b$, and by Neumaier [25], who improved a similar result of Gay [12], the radius of the limit x^∞ is still bounded by the righthand side of (10). Ris [29] observed that it is better to use Gauss-Seidel iteration in place of (11); we mentioned already that Gauss-Seidel iteration is indeed faster, and leads to a limit with smaller radius. For the special choice $C = \check{A}^{-1}$, Proposition 2.5 of Krawczyk and Neumaier [19] implies that $(CA)^G(Cb)$ has a still smaller radius; but the improvement is slight if $\beta \ll 1$.

To get least significant bit accuracy for thin A and b, Rump [33] proceeds slightly differently. He computes a sufficiently accurate approximation $\tilde{x}$ of $x^* = A^{-1}b$, constructs the smallest machine-representable interval x containing $\tilde{x}$, and uses the implication

$$Cb + (I - CA)x \subseteq \operatorname{int} x \Rightarrow A \text{ regular}, \ A^H b \subseteq x, \tag{12}$$

a consequence of Brouwer's fixpoint theorem, to check whether x really contains the solution x^*.

1.5 Options for sparse M-matrices. For sparse matrices, the inverse is generally full (Duff et al. [11]), and preconditioning is too time and/or space consuming. Also, Gaus-Seidel iteration is much too slow for most practical sparse problems. At present, no fast and reliable method is known for sparse linear interval equations whose coefficient matrix is neither an M-matrix nor diagonally dominant; the apparently quite general methods proposed by Rump [33] and Hahn et. al. [13] suffer from the overestimation problem of Gauss elimination (except for thin problems where the overestimation can be counteracted by using sufficiently accurate multiprecision approximations of the solution). We survey here the M-matrix case; all methods (but not the optimality results) generalize to H-matrices, and in particular to the diagonally dominant case. The methods known to me are:

a) Gauss elimination for matrices with small bandwidth or profile. This is almost optimal for sparse M-matrices, and the standard profile optimization algorithms can be used.

b) Gauss elimination coupled with the iteration (5); see Mayer [21], Schwandt [37]. If A_O is sparse and thin and $A = A_O - E$ is an M-matrix with $A \leqq A_O$, then the iteration converges to the hull (Neumaier [24]).

c) Iteration with incomplete factorizations; see Mayer [20]. This iteration is closely related to method b) but makes more flexible use of the zero pattern.

d) An interval Buneman algorithm; see Schwandt [36]. This algorithm applies to a restricted class of interval equations related to certain elliptic partial differential equations.

e) Aposteriori enclosure of a good approximate solution by Brouwer's fixpoint theorem (Rump [33]) or perturbation theorems (Hahn et al. [13]). The methods proposed are efficient only for special classes of coefficient matrices like M-matrices or diagonally dominant matrices; they behave badly e.g. on thin banded triangular H-matrices A with $A_{ii} = 3$, $A_{ii-1} = 4$, $A_{ii-2} = 5$ and $A_{ik} = 0$ for $k \neq i,i-1,i-2$, especially when the righthand side is not thin.

f) Modified Gauss-Seidel iteration. The iteration starts with the initial enclosure $x^0 := [-u,u]\,\|b\|_v$ from (7) and uses an accompanying approximate iteration $\tilde{x}^\ell$ ($\ell = 1,2,\ldots$) converging to $\check{A}^{-1}\check{b}$. A sequence of enclosures x^ℓ for $A^H b$ is found as

$$\left.\begin{aligned} x^{\ell+1/2} &:= \Gamma(A,b,x^\ell) \\ x^{\ell+1} &:= x^{\ell+1/2} \cap (\tilde{x}^{\ell+1} + [-u,u]\,\|b - A\tilde{x}^{\ell+1}\|_v) \end{aligned}\right\} (\ell = 0,1,2,\ldots) \quad (13)$$

This iteration converges for M-matrices A to the hull $A^H b$ independently of the choice of the "forcing sequence" $\tilde{x}^\ell$; however, the convergence speed is at least that of the forcing sequence. Thus, any fast-converging iteration scheme for the approximate solution of sparse linear equations can be used to speed up the process. It is also possible to use only one or two steps of (13) aposteriori, i.e. with a good approximation $\tilde{x}^1$ to $\check{A}^{-1}b$.

2. The rectangular case

In this section we treat the case of a rectangular coefficient matrix. We restrict ourselves to the overdetermined case (i.e. $m > n$); for the underdetermined case cf. Rump [33]. The problem was posed first by Jahn [16] who suggested Gauss elimination to solve the equation; however, by the same reasons as for the square case this is reliable only for $n = 2$.

2.1 The least squares hull. All other authors concerned with interval methods for overdetermined systems replace the problem of finding $A^H b$ by that of finding the least squares hull

$$A^L b := \square\{\tilde{x} \in \mathbb{R}^n \mid \tilde{A}^T\tilde{A}\tilde{x} = \tilde{A}^T\tilde{b} \text{ for some } \tilde{A} \in A,\ \tilde{b} \in b\}, \tag{14}$$

and reduce this problem to the square case by observing that

$$A^H b \subseteq A^L b \subseteq x \text{ , where } \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix}^H \begin{pmatrix} b \\ 0 \end{pmatrix} \tag{15}$$

(Spellucci and Krier [39], Rump [33]) or

$$A^H b \subseteq A^L b \subseteq (A^T A)^H (A^T b) \tag{16}$$

(Deif [10]). Unless one has specific reasons to work with $A^L b$ in place of $A^H b$, the approach via (15), (16) is not recommended since $A^L b$ depends very sensitively on scaling and since $\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix}$ and $A^T A$ are generally much more ill-conditioned than A (squared condition number).

In the thin case, where usually $A^H b = \emptyset$ and the original problem (1) makes no sense, $A^L b$ can often be computed in spite of ill-conditioning by using a sufficiently accurate approximate least squares solution $\tilde{x}$ and a residual form of (15); conditioning problems are reduced by the use of a highly precise arithmetic. See Rump [33].

2.2 Preconditioning. With an approximation C of a pseudo inverse of $\check{A}$ one gets

$$A^H b \subseteq (CA)^H (Cb). \tag{17}$$

If the rank of $\check{A}$ is n and $\rho(A)$ is sufficiently small then CA is a square n×n-matrix, almost the identity. Therefore, the methods discussed in Section 1.4 can be applied to enclose the righthand side of (17). C is computed in a stable way as $C = R^{-1}Q^T D^{-1}$ from an orthogonal decomposition of the diagonally scaled matrix $D\check{A} = QR$; here $D \in \mathbb{R}^{m\times n}$ is the diagonal scaling matrix, $R \in \mathbb{R}^{n\times n}$ is upper triangular, and $Q \in \mathbb{R}^{m\times n}$ consists of the first n columns of a square orthogonal m×m-matrix (Q,Q').

Since the righthand side of (17) is defined even if the system (1) is inconsistant, a consistency check is useful. By (2), a sufficient condtion for consistency is $A\tilde{x} \cap b \neq \emptyset$ for an approximate solution $\tilde{x}$; on the other hand,

$$A^H b \subseteq x,\ (a^T A)x \cap a^T b = \emptyset \Rightarrow A^H b = \emptyset \qquad (18)$$

holds for all $a \in \mathbb{R}^m$. If inconsistency of (18) is suspected one should check (18) for several choices of a; suitable vectors are e.g. the columns of $D^{-1}Q'$.

References.

1. G. Alefeld, Das symmetrische Einzelschrittverfahren bei linearen Gleichungen mit Intervallen als Koeffizienten, Computing 18, 329-340 (1977).

2. G. Alefeld, Über die Durchführbarkeit des Gaußschen Algorithmus bei Gleichungen mit Intervallen als Koeffizienten, Computing Suppl. 1, 15-19 (1977).

3. G. Alefeld, Intervallanalytische Methoden bei nichtlinearen Gleichungen, Jahrbuch Überblicke Mathematik 1979, (ed. S.D. Chatterji et al.), Bibl. Inst.,Mannheim-Wien-Zürich, 63-78 (1979).

4. N. Apostolatos und U. Kulisch, Grundzüge einer Intervallrechnung für Matrizen und einige Anwendungen, Elektron. Rechenanlagen 10, 73-83 (1968).

5. W. Barth und E. Nuding, Optimale Lösung von Intervallgleichungssystemen, Computing 12, 117-125 (1974).

6. H. Beeck, Über Struktur und Abschätzungen der Lösungsmenge von linearen Gleichungssystemen mit Intervallkoeffizienten, Computing 10, 231-244 (1972).

7. H. Beeck, Zur scharfen Außenabschätzung der Lösungsmenge bei linearen Intervallgleichungssystemen, Z. Angew. Math. Mech. 54, T208-T209 (1974).

8. H. Beeck, Zur Problematik der Hüllenbestimmung von Intervallgleichungssystemen, 'Interval Mathematics', (ed. K. Nickel), Lecture Notes in Computer Science 29, Springer Verlag, 150-159 (1975).

9. H. Cornelius, Untersuchungen zu einem intervallarithmetischen Iterationsverfahren mit Anwendungen auf eine Klasse nichtlinearer Gleichungssysteme, Dissertation, Techn. Univ. Berlin (1981).

10. A. S. Deif, to be published.

11. I. S. Duff, A. M. Erisman, C. W. Gear, and J. K. Reid, Some remarks on inverses of sparse matrices, Techn. Memorandum 51, Math. Comp. Sci. Div., Argonne Nat. Lab., Argonne, Illinois (1985).

12. D. M. Gay, Solving interval linear equations, SIAM J. Numer. Anal. 19, 858-870 (1982).

13. W. Hahn, K. Mohr, and U. Schauer, Some techniques for solving linear equation systems with guarantee, Computing 34, 375-379 (1985).

14. E. Hansen and S. Sengupta, Bounding solutions of systems of equations using interval analysis, BIT 21, 203-211 (1981).

15. E. Hansen and R. Smith, Interval arithmetic in matrix computations, Part II, SIAM J. Numer. Anal. 4, 1-9 (1967).

16. K.-U. Jahn, Eine Theorie der Gleichungssysteme mit Intervallkoeffizienten, Z. Angew. Math. Mech. 54, 405-412 (1974).

17. G. Kopp, Die numerische Behandlung von reellen linearen Gleichungssystemen mit Fehlererfassung für M-Matrizen sowie für diagonaldominante und invers-isotone Matrizen, Diplomarbeit, Inst. f. Prakt. Math. Univ. Karlsruhe (1976).

18. R. Krawczyk Newton-Algorithmen zur Bestimmung von Nullstellen mit Fehlerschranken, Computing 4, 187-201 (1969).

19. R. Krawczyk and A. Neumaier, Interval Newton operators for function strips, Freiburger Intervall-Berichte 85(7), 1-34 (1985).

20. G. Mayer, Enclosing the solution set of linear systems with inaccurate data by iterative methods based on incomplete LU-decompositions, Computing 35, 189-206 (1985).

21. G. Mayer, Comparison theorems for an iterative method based on strong splittings, to appear.

22. O. Mayer, Über intervallmäßige Iterationsverfahren bei linearen Gleichungssystemen und allgemeineren Intervallgleichungssystemen, Z. Angew. Math. Mech. 51, 117-124 (1971).

23. A. Neumaier, New techniques for the analysis of linear interval equations, Linear Algebra Appl. 58, 273-325 (1984).

24. A. Neumaier, Further results on linear interval equations, Freiburger Intervall-Berichte 85(4), 37-72 (1985).

25. A. Neumaier, Overestimation in linear interval equations, Freiburger Intervall-Berichte 85(4), 75-91 (1985).

26. W. Oettli and W. Prager, Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides, Numer. Math. 6, 405-409 (1964).

27. K. Reichmann, Ein hinreichendes Kriterium für die Durchführbarkeit des Intervall-Gauß-Algorithmus bei Intervall-Hessenbergmatrizen ohne Pivotsuche, Z. Angew. Math. Mech. 59, 373-379 (1979).

28. K. Reichmann, Abbruch beim Intervall-Gauß-Algorithmus, Computing 22, 355-361 (1979).

29. F. N. Ris, Interval analysis and applications to linear algebra, D. Phil. Thesis, Oxford (1972).

30. J. Rohn, An algorithm for solving interval linear systems and inverting interval matrices, Freiburger Intervall-Berichte 82(5), 23-36 (1982).

31. J. Rohn, Solving interval linear systems; Proofs to 'Solving interval linear systems'; Interval linear systems, Freiburger Intervall-Berichte 84(7), 1-14, 17-30, 33-58 (1984).

32. J. Rohn, Some results on interval linear systems, Freiburger Intervall-Berichte 85(4), 93-116 (1985).

33. S. M. Rump, Solving algebraic problems with high accuracy, 'A new approach to scientific computation', (ed. U. W. Kulisch and W. L. Miranker), Academic Press, New York, 51-120 (1983).

34. F. Schätzle, Überschätzung beim Gauß-Algorithmus für lineare Intervallgleichungssysteme, Freiburger Intervall-Berichte 84(3), (1984).

35. H. Schwandt, Schnelle fast global konvergente Verfahren für die Fünf-Punkt-Diskretisierung der Poissongleichung mit Dirichletschen Randbedingungen auf Rechteckgebieten, Dissertation, Techn. Univ. Berlin (1981).

36. H. Schwandt, An interval arithmetic approach for the construction of an almost globally convergent method for the solution of the nonlinear Poisson equation on the unit square, SIAM J. Sci. Statist. Comput. 5, 427-452 (1984).

37. H. Schwandt, Krawczyk-like algorithmus for the solution of systems of nonlinear equations, SIAM J. Numer. Anal. 22, 792-810 (1985).

38. J. M. Shearer and M. A. Wolfe, Some algorithms for the solution of a class of nonlinear algebraic equations, Computing 35,63-72 (1985).

39. P. Spellucci und N. Krier, Untersuchungen der Grenzgenauigkeit von Algorithmen zur Auflösung linearer Gleichungssysteme mit Fehlererfassung, 'Interval Mathematics', (ed. K. Nickel), Lecture Notes in Computer Science 29, Springer Verlag, 288-297 (1975).

40. P. Wongwises, Experimentelle Untersuchungen zur numerischen Auflösung von linearen Gleichungssystemen mit Fehlererfassung, 'Interval Mathematics', (ed. K. Nickel), Lecture Notes in Computer Science 29, Springer Verlag, 316-325 (1975).