ARITHMETIC OF CIRCULAR RINGS

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1. INTRODUCTION

Circular arithmetic, introduced by I. Gargantini and P. Henrici [2] as an extension of the complex arithmetic, provided the formulation of methods for solving some problems of computational complex analysis (e.g. the inclusion of the polynomial complex zeros [2], [3], circular approximation of the closed regions in the complex plane [1], [4], [6], [8], the evaluations of complex functions over a disk as an argument [4], [5], [6], [8], [9], etc.). Applying these methods, sometimes a problem of evaluation with the disks which contain the origin arises (for example, inversion of a disk Z, evaluation of the complex functions $z \mapsto \ln z$, $z \mapsto z^{1/k}$ over a disk Z, where $0 \in Z$). In some cases, this problem can be overcome by evaluation with the annulus $\{z : r \leq |z-c| \leq R\}$ instead of the disk $\{z : |z-c| \leq R\}$ if the origin is contained in the internal disk $\{z : |z-c| \leq r\}$. In this way, an isolation of zero has been done. Besides, in some cases, a set of points Ω in the complex plane (e.g. an opened or closed curve which is characterized by the circularity) can be enclosed by an annulus, say Z. For a given analytic function f, this inclusion enables to consider in the sequel an *annular* function F such that

$$F(Z) \supseteq f(Z) = \{f(z) : z \in Z\} \supseteq f(\Omega) \quad (\Omega \subseteq Z).$$

Note that T.J. Rivlin [7] used the annulli for assignation of a measure of circularity to a given compact set in the plane. His approach is to determine the best annulus which contains the given set according to the size of the annulus.

In this paper we shall pay attention only to the problem of defining the basic arithmetic operations with the circular rings. Further, we shall point out some possibilities to use the arithmetic of annulli, which is an extension of circular arithmetic. Note that this generalization carries some disadventages, for example, more complicated arithmetic operations and reduction of these operations to the operations in circular arithmetic in some cases.

2. ARITHMETIC OPERATIONS

A set $Z = \{z : r \leq |z-c| \leq R,\ c \in C,\ 0 \leq r \leq R\}$, denoted by $Z = \{c; (r,R)\}$, will be called a *circular ring* or *annulus*. Here c is the center, r and R are internal and external radii of the annulus Z, respectively. The set of all circular rings will be denoted by $A(C)$. In the special case, for $r = 0$, the annulus $Z = \{c\ ;\ (0,R)\}$ reduces to the disk $Z = \{z: |z-c| \leq R\}$, denoted shorter by $Z = \{c\ ; R\}$.

Sometimes, an annulus whose center is at the origin, will be marked by $\overset{\circ}{Z}$, that is

$$\overset{\circ}{Z} = \{0\ ;\ (r,R)\} = \{\rho e^{i\phi}:\ r \leq \rho \leq R,\ 0 \leq \phi \leq 2\pi\}.$$

An annulus $Z = \{c\ ;\ (r,R)\}$ can be presented in the form

$$Z = \{c\ ;\ (r,R)\} = c + \overset{\circ}{Z}\ , \tag{1}$$

because of

$$Z = \{c + \rho e^{i\phi}:\ r \leq \rho \leq R,\ 0 \leq \phi \leq 2\pi\}$$
$$= \{c\} + \{\rho e^{i\phi}:\ r \leq \rho \leq R,\ 0 \leq \phi \leq 2\pi\}.$$

ADDITION OF A SCALAR AND AN ANNULUS :

Since

$$w + \{c\ ;\ (r,R)\} = \{w + z:\ r \leq |z-c| \leq R\} = \{z:\ r \leq |z-(w+c)| \leq R\},$$

we have

$$w + \{c\ ;\ (r,R)\} = \{w + c\ ;\ (r,R)\}. \tag{2}$$

MULTIPLICATION OF A SCALAR AND AN ANNULUS :

Let $w \in C$, $\overset{\circ}{Z} = \{0\ ;\ (r,R)\}$ and $Z = \{c\ ;\ (r,R)\}$. First, we have

$$w\cdot\overset{\circ}{Z} = w\cdot\{0\ ;\ (r,R)\} = w\cdot\{z = \rho e^{i\phi}:\ r \leq \rho \leq R,\ 0 \leq \phi \leq 2\pi\}$$
$$= \{z' = |w|\rho e^{i(\phi+\arg w)}:\ r \leq \rho \leq R,\ 0 \leq \phi \leq 2\pi\}$$
$$= \{z' = \rho' e^{i\phi'}:\ |w|r \leq \rho' \leq |w|R,\ 0 \leq \phi' \leq 2\pi\},$$

wherefrom

$$w\cdot\overset{\circ}{Z} = \{0\ ;\ (|w|r, |w|R)\}. \tag{3}$$

On the basis of (2) and (3) we find

$$w\cdot Z = w\cdot\{c\ ;\ (r,R)\} = w\cdot(c + \overset{\circ}{Z}) = wc + w\cdot\overset{\circ}{Z}$$
$$= wc + \{0\ ;\ (|w|r, |w|R)\} = \{wc\ ;\ (|w|r, |w|R)\},$$

that is,

$$w\cdot\{c\ ;\ (r,R)\} = \{wc\ ;\ (|w|r,|w|R)\}. \qquad (4)$$

INCLUSION AND DISJUNCTION OF THE ANNULLI:

Let $Z_i = \{c_i\ ;\ (r_i,R_i)\} \in A(C)$, $i=1,2$. The annulus Z_2 *contains* the annulus Z_1, denoted by $Z_1 \subseteq Z_2$, if and only if

$$\Big((R_2 > R_1)\wedge(r_1 > r_2)\wedge(|c_2-c_1| < \min\{r_1-r_2, R_2-R_1\})\Big)\vee\Big(R_2-R_1 > |c_1| > r_2+R_1\Big).$$

The annulli Z_1 and Z_2 are *disjoint* (i.e. $Z_1 \cap Z_2 = \emptyset$) if and only if one of the following conditions is valid:

$$\Big(|c_2-c_1| > R_1+R_2\Big)\vee\Big((r_2 > R_1)\wedge(|c_2-c_1| < r_2-R_1)\Big)\vee\Big((r_1 > R_2)\wedge(|c_2-c_1| < r_1-R_2)\Big).$$

ADDITION AND SUBSTRACTION:

Let $\overset{\circ}{Z}_i = \{0\ ;\ (r_i,R_i)\}$, $i=1,2$. Then

$$\overset{\circ}{Z}_1 + \overset{\circ}{Z}_2 = \{\rho_1 e^{i\phi_1} + \rho_2 e^{i\phi_2} : r_i \leq \rho_i \leq R_i,\ 0 \leq \phi_i \leq 2\pi,\ i=1,2\}$$

$$= \{0\ ;\ (r,R)\},$$

where

$$r = \min_{\substack{\rho_1,\rho_2\\ \phi_1,\phi_2}} |\rho_1 e^{i\phi_1} + \rho_2 e^{i\phi_2}|,$$

$$R = \max_{\substack{\rho_1,\rho_2\\ \phi_1,\phi_2}} |\rho_1 e^{i\phi_1} + \rho_2 e^{i\phi_2}|.$$

It is sufficient to take $\phi_i \in [0,\pi]$ $(i=1,2)$. Obviously,

$$R = \max_{r_1 \leq \rho_1 \leq R_1} \rho_1 + \max_{r_2 \leq \rho_2 \leq R_2} \rho_2 = R_1 + R_2,$$

$$r = \min_{\substack{r_1 \leq \rho_1 \leq R_1\\ r_2 \leq \rho_2 \leq R_2}} |\rho_1 - \rho_2|.$$

The value of r is given by

$$r = \begin{cases} r_2 - R_1, & \text{if } r_2 > R_1,\\ r_1 - R_2, & \text{if } r_1 > R_2,\\ 0, & \text{otherwise, that is, if } \overset{\circ}{Z}_1 \cap \overset{\circ}{Z}_2 \neq \emptyset. \end{cases} \qquad (5)$$

Now, we have

$$\{0\ ;\ (r_1,R_2)\} + \{0\ ;\ (r_2,R_2)\} = \{0\ ;\ (r,R_1+R_2)\}, \tag{6}$$

where r is defined by (5).

According to (6) it follows

$$Z_1 + Z_2 = \{c_1\ ;\ (r_1,R_1)\} + \{c_2\ ;\ (r_2,R_2)\}$$
$$= (c_1 + \overset{o}{Z}_1) + (c_2 + \overset{o}{Z}_2) = c_1 + c_2 + (\overset{o}{Z}_1 + \overset{o}{Z}_2),$$

or

$$Z_1 + Z_2 := \{c_1 + c_2\ ;\ (r,R_1+R_2)\}. \tag{7}$$

Thus, *the addition* in the set A(C) is defined by (7), where r is given by (5).

It is easy to prove that for the addition of the annulli the following is valid:

$$Z_1 + Z_2 = Z_2 + Z_1 \qquad (\textit{commutativity}),$$
$$(Z_1 + Z_2) + Z_3 = Z_1 + (Z_2 + Z_3) \qquad (\textit{associativity}).$$

Let $Z_i = \{c_i\ ;\ (r_i,R_i)\}$ (i=1,2,3). The final result of the addition of three annulli we denote by $Z_1 + Z_2 + Z_3$, and it is given by

$$Z_1 + Z_2 + Z_3 = \{c_1 + c_2 + c_3\ ;\ (r,R_1 + R_2 + R_3)\}, \tag{8}$$

where

$$r = \begin{cases} r_1 - R_2 - R_3, & \text{if } r_1 > R_2 + R_3, \\ r_2 - R_1 - R_3, & \text{if } r_2 > R_1 + R_3, \\ r_3 - R_1 - R_2, & \text{if } r_3 > R_1 + R_2, \\ 0, & \text{otherwise.} \end{cases}$$

The substraction of the annulli is defined using (3) and (7):

$$\{c_1\ ;\ (r_1,R_1)\} - \{c_2\ ;\ (r_2,R_2)\} := \{c_1\ ;\ (r_1,R_1)\} + \{-c_2\ ;\ (r_2,R_2)\}$$
$$= \{c_1 - c_2\ ;\ (r,R_1 + R_2)\}, \tag{9}$$

where, again, r is given by (5).

MULTIPLICATION:

The product of two annulli Z_1Z_2 is not an annulus in general. For this reason, an extended set $Z_1 \otimes Z_2$ in the form of an annulus, such that $Z_1 \otimes Z_2 \supseteq Z_1Z_2$, is introduced. The symbol $\otimes$ denotes the multiplication in the set A(C).

The product of the annulli $\{0;(r_1,R_1)\}$ and $\{0;(r_2,R_2)\}$ is given by

$$\{0;(r_1,R_1)\} \otimes \{0;(r_2,R_2)\} = \{\rho_1 e^{i\phi_1}: r_1 \leq \rho_1 \leq R_1,\ 0 \leq \phi_1 \leq 2\pi\} \otimes \{\rho_2 e^{i\phi_2}: r_2 \leq \phi_2 \leq R_2,\ 0 \leq \phi_2 \leq 2\pi\}$$

$$= \{\rho e^{i\phi}: r_1 r_2 \leq \rho \leq R_1 R_2,\ 0 \leq \phi \leq 2\pi\}$$

$$= \{0\ ;\ (r_1 r_2, R_1 R_2)\},$$

where $\rho = \rho_1\rho_2$ and $\phi = \phi_1+\phi_2$. Accordingly,

$$\{0\ ;\ (r_1,R_1)\} \otimes \{0\ ;\ (r_2,R_2)\} = \{0\ ;\ (r_1 r_2, R_1 R_2)\}. \tag{10}$$

It is obvious that

$$\mathring{Z}_1 \otimes \mathring{Z}_2 = \mathring{Z}_1 \mathring{Z}_2 = \{z_1 z_2 : z_1 \in \mathring{Z}_1, z_2 \in \mathring{Z}_2\}.$$

By induction we show that

$$\prod_{k=1}^{n} \mathring{Z}_k = \left\{0\ ;\ \left(\prod_{k=1}^{n} r_k\ ,\ \prod_{k=1}^{n} R_k\right)\right\}$$

is valid, wherefrom, in a special case, we have

$$\{0\ ;\ (r,R)\}^n = \{0\ ;\ (r^n,R^n)\}\ . \tag{11}$$

Let $Z_i = \{c_i\ ;\ (r_i,R_i)\} \in A(C)$ $(i=1,2)$. According to (10) we find

$$Z_1 Z_2 = \{c_1\ ;\ (r_1,R_1)\}\cdot\{c_2\ ;\ (r_2,R_2)\}$$

$$= (c_1 + \{0;(r_1,R_1)\})\cdot(c_2 + \{0;(r_2,R_2)\})$$

$$\subseteq c_1 c_2 + c_2\cdot\{0;(r_1,R_1)\} + c_1\cdot\{0;(r_2,R_2)\} + \{0;(r_1,R_1)\}\cdot\{0;(r_2,R_2)\}.$$

Applying (4) and (10), we obtain that

$$Z_1 Z_2 \subseteq c_1 c_2 + \{0;(|c_2|r_1,|c_2|R_1)\} + \{0;(|c_1|r_2,|c_1|R_2)\} + \{0;(r_1 r_2, R_1 R_2)\}.$$

In view of (8) and (9), it follows

$$Z_1 Z_2 \subseteq c_1 c_2 + \{0\ ;\ (r,R)\} = \{c_1 c_2\ ;\ (r,R)\},$$

where

$$R = |c_2|R_1 + |c_1|R_2 + R_1 R_2 \tag{12}$$

and

$$r = \begin{cases} r_1r_2 - |c_2|R_1 - |c_1|R_2, & \text{if } r_1r_2 > |c_2|R_1 + |c_1|R_2 \\ |c_1|r_2 - |c_2|R_1 - R_1R_2, & \text{if } |c_1|r_2 > |c_2|R_1 + R_1R_2 \\ |c_2|r_1 - |c_1|R_2 - R_1R_2, & \text{if } |c_2|r_1 > |c_1|R_2 + R_1R_2 \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

For the product $Z_1 \otimes Z_2$ of the annulli Z_1 and Z_2 we shall adopt the extended set $\{c_1c_2 \,;\, (r,R)\}$, that is

$$Z_1Z_2 \subseteq Z_1 \otimes Z_2 := \{c_1c_2 \;;\; (r,R)\}, \tag{14}$$

where r and R are given by (12) and (13) respectively.

It is easy to show that four cases in (13) are disjoint. Let us show, for example, that the inequalities

$$|c_1|r_2 > |c_2|R_1 + R_1R_2,$$
$$|c_2|r_1 > |c_1|R_2 + R_1R_2,$$

can not be valid simultaneously. Rewrite the above inequalities in the form

$$y = |c_1|r_2 - |c_2|R_1 > R_1R_2, \tag{*}$$
$$|c_2|r_1 - |c_1|R_2 > R_1R_2, \tag{**}$$

and put

$$R_1 = r_1 + \varepsilon_1, \quad R_2 = r_2 + \varepsilon_2 \quad (\varepsilon_1, \varepsilon_2 > 0).$$

Suppose that (*) holds. Then $y > 0$, while the left-hand side of (**) becomes

$$\begin{aligned} |c_2|r_1 - |c_1|R_2 &= |c_2|(R_1-\varepsilon_1) - |c_1|(r_2+\varepsilon_2) \\ &= |c_2|R_1 - |c_2|\varepsilon_1 - |c_1|r_2 - |c_1|\varepsilon_2 \\ &= -y - |c_2|\varepsilon_1 - |c_1|\varepsilon_2 < 0. \end{aligned}$$

Thus, if (*) is valid, then (**) does not hold.

Note that, in the case when Z_1 and Z_2 are disks, that is, $Z_1 = \{c_1 \,;\, (0,R_1)\} = \{c_1 \,;\, R_1\}$ and $Z_2 = \{c_2 \,;\, (0,R_2)\} = \{c_2 \,;\, R_2\}$, the definition (14) for the product of annulli reduces to the definition for the product of disks introduced by I. Gargantini and P. Henrici [2].

Since the exchange of the indices at (12) and (13) (i.e. $1 \to 2$ and $2 \to 1$) does not cause any modification in the expressions for R and r, we conclude that

$$Z_1 \otimes Z_2 = Z_2 \otimes Z_1,$$

i.e., the product of the annulli, introduced by (14), is *commutative*.

INVERSION OF ANNULUS:

Defining the inversion of an annulus, we shall use the formula for the inversion of a disk $\{c\,;r\}$ which does not contain the origin,

$$\{c\,;r\}^{-1} = \left\{\frac{\bar{c}}{|c|^2 - r^2}\,;\,\frac{r}{|c|^2 - r^2}\right\}. \tag{15}$$

An inverse annulus Z^{-1}, where $Z = \{c;(r,R)\}$, will be defined in the case when $0 \notin Z$, i.e. if $|c| > R$ or $|c| < r$ holds (which is equivalent to $0 \notin \{c\,;R\}$ and $0 \in \{c\,;r\}$, respectively). Defining the inverse annulus we shall use the inverse disks

$$\{c\,;R\}^{-1} = \{w_1\,;\rho_1\}, \qquad \{c\,;r\}^{-1} = \{w_2\,;\rho_2\},$$

where, on the basis of (15),

$$w_1 = \frac{\bar{c}}{|c|^2 - R^2}, \quad \rho_1 = \frac{R}{|c|^2 - R^2}, \quad w_2 = \frac{\bar{c}}{|c|^2 - r^2}, \quad \rho_2 = \frac{r}{|c|^2 - r^2}.$$

We observe that the centers w_1 and w_2 are not overlapping, which means that the circles $\Gamma_1 = \{w\colon |w-w_1| = \rho_1\}$ and $\Gamma_2 = \{w\colon |w-w_2| = \rho_2\}$ are not concentric (see Fig. 1 and 2). For this reason, an extension of the exact set $\{z^{-1}\colon r \leq |z-c| \leq R\}$ must be taken to be the inverse annulus Z^{-1}. The extension has to be performed so that the implication

$$0 \notin Z \quad \Rightarrow \quad 0 \notin Z^{-1}$$

holds, which is of the essential interest in executing the arithmetic operations in the set $A(C)$. We shall distinguish two cases: $0 \notin \{c;R\}$ and $0 \in \{c;r\}$.

a) $0 \notin \{c\,;R\}$

In this case the origin is outside of the external circle and, thus, $|c| > R$ is valid. The circumferences Γ_1 and Γ_2 are not concentric so that we shall construct the inverse annulus Z^{-1} by extending the exact range $\{z^{-1}\colon r \leq |z-c| \leq R\}$ over the internal disk $\{c\,;r\}$ taking the point w_1 for the center of the inverse annulus Z^{-1}. In this manner the implication $0 \notin Z \Rightarrow 0 \notin Z^{-1}$ is provided (because the condition $0 \notin \{c\,;R\}$ provides that $0 \notin \{c\,;R\}^{-1}$).

From Fig. 1 we have

$$Z^{-1} := \{w_1\,;\,(\max\{0\,,\rho_2 - |w_1-w_2|\}\,,\rho_1)\}$$
$$= \frac{1}{|c|^2 - R^2}\left\{\bar{c}\,;\left(\max\left\{0\,,\frac{r|c| - R^2}{|c| - r}\right\},\ R\right)\right\}.$$

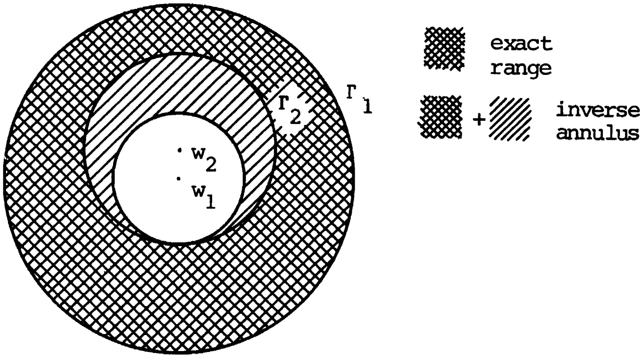

Fig. 1 Inverse annulus: the case $0 \notin \{c\,;R\}$

b) $0 \in \{c\,;r\}$

In this case we have $|c| < r$ and

$$\{c\,;R\}^{-1} = \{w:\ |w-w_1| \geqq \rho_1\}, \qquad \{c\,;r\}^{-1} = \{w:\ |w-w_2| \geqq \rho_2\},$$

where

$$w_1 = \frac{\bar{c}}{|c|^2 - R^2}, \quad \rho_1 = \frac{R}{R^2 - |c|^2}, \quad w_2 = \frac{\bar{c}}{|c|^2 - r^2}, \quad \rho_2 = \frac{r}{r^2 - |c|^2}.$$

The origin belongs to the interior of the smaller circle $\Gamma_1 = \{w:\ |w - w_1| = \rho_1\}$ because of $|w_1| < \rho_1$. For this reason the extension of the exact range will be performed outside of the circle Γ_2, taking the point w_1 for the center of inverse annulus.

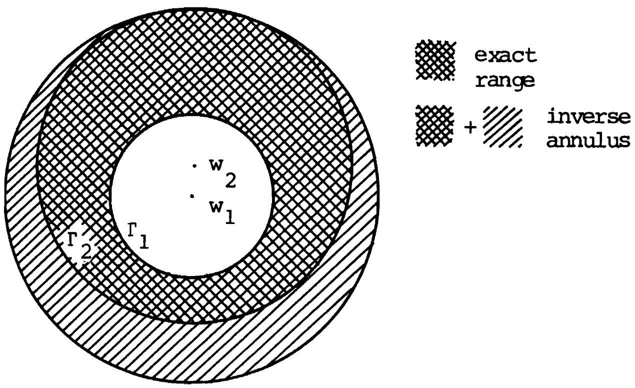

Fig. 2 Inverse annulus: the case $0 \in \{c\,;r\}$

From Fig. 2 we find

$$Z^{-1}: = \{w_1 \; ; \; (\rho_1 \, , \, |w_2 - w_1| + \rho_2)\}$$
$$= \frac{1}{R^2 - |c|^2} \left\{ -\bar{c} \; ; \; \left(R \, , \, \frac{R^2 - r|c|}{r - |c|} \right) \right\}.$$

Thus, the inverse annulus Z^{-1} $(0 \notin Z)$ is defined by

$$Z^{-1}: = \begin{cases} \dfrac{1}{|c|^2 - R^2} \left\{ \bar{c} \; ; \left(\max\left\{ 0 \, , \dfrac{r|c| - R^2}{|c| - r} \right\}, R \right) \right\} & \text{if } |c| > R \, , \\ \dfrac{1}{R^2 - |c|^2} \left\{ -\bar{c} \; ; \left(R \, , \dfrac{R^2 - r|c|}{r - |c|} \right) \right\} & \text{if } |c| < r \, . \end{cases} \tag{16}$$

DIVISION:

Let the symbol $\oslash$ denotes the division in A(C). Using the definitions (14) and (16) for the multiplication and the inversion, we define the operation of division in A(C) as follows:

$$Z_1 \oslash Z_2: = Z_1 \otimes Z_2^{-1} \quad (0 \notin Z_2). \tag{17}$$

3. ANNULAR FUNCTIONS

Let f be an analytic function defined on the region D in the complex plane, and let Z be an annulus contained in D. The set $f(Z) = \{f(z): z \in Z\}$ is not an annulus in general. In order to use the arithmetic of annulli, a necessity arises for introducing an annular function $F: G \to H$ $(G,H \subset A(D))$ such that the following is valid:

$$\begin{aligned} F(Z) &\supseteq f(Z) \quad \text{for all } Z \in A(D), \\ F(z) &= f(z) \quad \text{for all } z \in Z. \end{aligned} \tag{18}$$

The function F such that (18) is satisfied is called an *annular inclusive extension* of f.

Let Ω be a set of points in the complex plane such that it can be suitably „enclosed" by an annulus Z, that is, $\Omega \subseteq Z$. Assume that f is a complex function such that the set f(Z) is a closed region and let F be an annular inclusive extension of f. Since $f(\Omega) \subseteq f(Z) \subseteq F(Z)$, we can consider the annulus F(Z) (for a given Z), which presents an annular approximation of the set $f(\Omega)$, instead of the image $f(\Omega)$. The approximation („covering") of the set Ω by an annulus Z is of special interest if the set Ω is characterized by a certain circularity. Ω can be also some opened or closed curve, $\Omega = \{\omega(t): \omega(t) = u(t) + i\,v(t) \, , \, t \in (\alpha,\beta)\}$, as it has been mentioned in the beginning. For example, an ellipse given by $\omega(t) = a\cos t + i\,b\sin t$ $(t \in [0,2\pi),\ a > b)$, can be bounded by an annulus $\overset{\circ}{Z} = \{0\ ;\ (b,a)\}$. This approximation is better if the quotient $\frac{a}{b}$ is closer to 1.

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