INNER SOLUTIONS OF LINEAR INTERVAL SYSTEMS

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A vector $x \in R^n$ is called an inner solution of a system of linear interval equations $A^I x = b^I$ ($A^I = [\underline{A},\overline{A}] = [A_c - \Delta,\ A_c + \Delta]$ of size m×n, $b^I = [\underline{b},\overline{b}] = [b_c - \delta,\ b_c + \delta]$) if $Ax \in b^I$ for each $A \in A^I$ (for a motivation, see [1]). Denote by X_i the set of all inner solutions. We have this characterization:

Theorem. $x \in X_i$ if and only if $x = x_1 - x_2$, where x_1, x_2 is a solution to the system of linear inequalities

$$\begin{aligned} \overline{A}x_1 - \underline{A}x_2 &\leqq \overline{b} \\ -\underline{A}x_1 + \overline{A}x_2 &\leqq -\underline{b} \\ x_1 \geqq 0, \quad x_2 &\geqq 0. \end{aligned} \qquad \text{(S)}$$

Proof. Due to Oettli-Prager theorem, $\{Ax;\ A \in A^I\} = [A_c x - \Delta|x|,\ A_c x + \Delta|x|]$. "Only if": Let $x \in X_i$, then $\underline{b} \leqq A_c x - \Delta|x|$ and $A_c x + \Delta|x| \leqq \overline{b}$; substituting $x = x^+ - x^-$, $|x| = x^+ - x^-$, we see that $x_1 = x^+$, $x_2 = x^-$ satisfy (S). "If": Let x_1, x_2 solve (S); define $d \in R^n$ by $d_j = \min\{x_{1j}, x_{2j}\}\ \forall\ j$, then $d \geqq 0$ for $x = x_1 - x_2$ we have $x^+ = x_1 - d$, $x^- = x_2 - d$, hence $A_c x + \Delta|x| = \overline{A}x_1 - \underline{A}x_2 - 2\Delta d \leqq \overline{b}$, similarly $A_c x - \Delta|x| \geqq \underline{b}$. Thus $[A_c x - \Delta|x|,\ A_c x + \Delta|x|] \subset b^I$, implying $x \in X_i$. ∎

As consequences, we obtain: (i) X_i is a convex polytope, (ii) each $x \in X_i$ satisfies $\Delta|x| \leqq \delta$ (by adding the first two inequalities in (S)), (iii) X_i is bounded if for each j there is a k with $\Delta_{kj} > 0$ (since then from (ii) follows $|x_j| \leqq \delta_k/\Delta_{kj}$), (iv) $X_i \neq \emptyset$ if and only if (S) has a solution, which can be tested by phase I of the simplex algorithm, (v) for $\underline{x}_j = \min\{x_j;\ x \in X_i\}$ we have $\underline{x}_j = \min\{(x_1 - x_2)_j;\ x_1, x_2 \text{ solve (S)}\}$, which is a linear programming problem (similarly for $\overline{x}_j = \max\ldots$), (vi) nonnegative inner solutions are described by $\overline{A}x \leq \overline{b}$, $-\underline{A}x \leq -\underline{b}$, $x \geq 0$, (vii) also, $X_i = \{x;\ |A_c x - b_c| \leq -\Delta|x| + \delta\}$ (observe the similarity with the Oettli-Prager result).

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Reference.

[1] Nuding, E.; Wilhelm, J.: Über Gleichungen und über Lösungen, ZAMM 52, T188 - T190 (1972).