EMBEDDING THEOREMS FOR CONES AND APPLICATIONS TO CLASSES OF CONVEX SETS OCCURRING IN INTERVAL MATHEMATICS

Klaus D. Schmidt

Seminar für Statistik, Universität Mannheim, A 5
6800 Mannheim, West Germany

ABSTRACT

This paper gives a survey of embedding theorems for cones and their application to classes of convex sets occurring in interval mathematics.

1. INTRODUCTION

In many situations, the investigation of set-valued maps can be reduced to the vector-valued case by applying embedding theorems for classes of convex sets. For example, Rådström's embedding theorem for the class of all nonempty, compact, convex subsets of a normed vector space has been used in the construction of the Debreu integral and in the proof of a law of large numbers for random sets, and there is some hope that such embedding theorems can also be used for proving fixed-point theorems for set-valued maps which are needed in interval mathematics and other areas like mathematical economics.

An interesting method for proving embedding theorems for classes of convex sets is that of Rådström [10] who first established the cone properties of the class of convex sets under consideration and then applied a general embedding theorem for cones to prove his embedding theorems for the class of all nonempty, compact, convex subsets of a normed vector space and for the class of all nonempty, closed, bounded, convex subsets of a reflexive Banach space. Rådström's method has also been used by Urbanski [15] who considered a more general situation, and it has quite recently been used by Fischer [1] who proved an embedding theorem for the class of all hypernorm balls of a hypernormed vector space which can be applied to norm balls and order intervals.

Since the value of such embedding theorems for classes of convex sets depends on the amount of information they provide on the embedding vector space and the embedding map, it is desirable to have embedding theorems which also reflect the inclusion of sets as an order relation as well as the relationship existing between the topological and order properties of the class of convex sets under consideration. For example, taking into account the inclusion of sets as an order relation has led to more informative versions of the embedding theorems proven by Rådström [10], Hörmander [2], and Fischer [1], and it has also led to new embedding theorems for the class of all order intervals of an (M-normed) vector lattice (with unit); see [13] and [14]. With regard to Rådström's method for proving embedding theorems for classes of convex sets, the results of [13] and [14] suggest a systematic study of embedding theorems for (topological) ordered cones.

The purpose of these notes is to give a survey of embedding theorems for cones and their application to classes of convex sets occuring in interval mathematics.

In Section 2, we present some known and several new embedding theorems for cones, ordered cones, topological cones, and topological ordered cones. As far as topological properties are concerned, we confine ourselves to the case where the topology is determined by a positively homogeneous translation-invariant metric. For applications in interval mathematics, this seems to be a reasonable restriction, as remarked by Ratschek [11].

In Section 3, we study the cone properties of norm balls, hypernorm balls, and order intervals. These classes of convex sets are always endowed with the Minkowski addition of sets, the usual multiplication of sets by positive scalars, and the inclusion of sets as an order relation, and the metrics under consideration are those of Hausdorff and Moore. Using the results of Section 3 and applying suitable embedding theorems for cones given in Section 2, it is not hard to establish embedding theorems for norm balls, hypernorm balls, and order intervals. For the brevity of the presentation in these notes, however, the formulation of the resulting embedding theorems must be left to the reader, but we remark that some of them may be found in [14]; see also [13] for intervals on the real line.

In Section 4, we indicate some further aspects of embedding theorems for classes of convex sets occuring in interval mathematics. In particular, we briefly discuss the relationship between quasilinear spaces and cones, we sketch the cone properties of order intervals with respect to an order relation which differs from the inclusion of sets but has the advantage of extending the order relation of the underlying vector lattice, and we also include some comments on concrete embedding theorems for order intervals.

For any details concerning ordered vector spaces and (normed) vector lattices, we refer to the books by Luxemburg and Zaanen [6] and by Schaefer [12].

2. EMBEDDING THEOREMS FOR CONES

In this section, we present some known and several new embedding theorems for cones, ordered cones, normed cones, and normed ordered cones. For the formulation of these embedding theorems, we have to introduce some new terminology for cones. Although some of the new definitions we introduce may be tentative, they are convenient for our purposes, and all of them are in accordance with the corresponding definitions for vector spaces. The proofs of the embedding theorems for cones are somewhat technical and lengthy and cannot be included in these notes. However, to give an idea of the proofs, we indicate the construction of the embedding vector space and the embedding map. Some further details may be found in the papers of Rådström [10] and Kaucher [4]; see also [13] and [14].

Cones

A cone (or semilinear space [1]) is a set $\mathbb{F}$ with a distinguished element $Z \in \mathbb{F}$ (called the zero element), a map $+ : \mathbb{F}\times\mathbb{F} \longrightarrow \mathbb{F}$ (called addition) satisfying $A+(B+C) = (A+B)+C$, $A+B = B+A$, and $A+Z = A$ for all $A, B, C \in \mathbb{F}$, and a map $\mathbb{R}_+\times\mathbb{F} \longrightarrow \mathbb{F}$ (called scalar multiplication) satisfying $\lambda(A+B) = \lambda A+\lambda B$, $(\lambda+\mu)A = \lambda A+\mu A$, $(\lambda\mu)A = \lambda(\mu A)$, $1A = A$, and $0A = Z$ for all $A, B \in \mathbb{F}$ and $\lambda, \mu \in \mathbb{R}_+$. If $\mathbb{F}$ is a cone with zero element Z, then the identity $\lambda Z = Z$ holds for all $\lambda \in \mathbb{R}_+$. A cone $\mathbb{F}$ has the cancellation property if $A = B$ holds for all $A, B \in \mathbb{F}$ satisfying $A+C = B+C$ for some $C \in \mathbb{F}$.

Let $\mathbb{F}$ be a cone having the cancellation property.
On $\mathbb{F}\times\mathbb{F}$, define an equivalence relation $\sim$ by letting

$$(A,B) \sim (C,D)$$

for all $(A,B), (C,D) \in \mathbb{F}\times\mathbb{F}$, and let

$$\langle A,B\rangle$$

denote the equivalence class containing (A,B).
Let $\mathbb{G}$ denote the collection of all equivalence classes of $\mathbb{F}\times\mathbb{F}$.

On $\mathbb{G}$, define addition $+ : \mathbb{G}\times\mathbb{G} \longrightarrow \mathbb{G}$ and scalar multiplication $\mathbb{R}\times\mathbb{G} \longrightarrow \mathbb{G}$ by letting

$$<A,B> + <C,D> \quad := \quad <A+C,B+D>$$

and

$$\alpha<A,B> \quad := \quad \begin{cases} < \alpha A \, , \, \alpha B > \; , & \text{if } \alpha \in \mathbb{R}_+ \\ <(-\alpha)B,(-\alpha)A> \; , & \text{otherwise} \end{cases}$$

for all $<A,B>, <C,D> \in \mathbb{G}$ and $\alpha \in \mathbb{R}$.
Furthermore, define a map $j : \mathbb{F} \longrightarrow \mathbb{G}$ by letting

$$j(A) \quad := \quad <A,Z>$$

for all $A \in \mathbb{F}$.
Then we have the following *basic embedding theorem* for cones, which is due to Rådström [10]:

2.1. Theorem.
Suppose $\mathbb{F}$ is a cone having the cancellation property.
Then $\mathbb{G}$ is a vector space satisfying $\mathbb{G} = j(\mathbb{F}) - j(\mathbb{F})$, and j is an injection which is additive and positively homogeneous.
In particular, the identities $<A,B> + <Z,Z> = <A,B>$ and $<A,B> + <B,A> = <Z,Z>$ hold for all $<A,B> \in \mathbb{G}$.

In the formulation of subsequent embedding theorems, we shall usually not repeat those properties of the embedding vector space $\mathbb{G}$ and the embedding map j which are evident from more general results.

Ordered Cones

An *ordered cone* is a cone $\mathbb{F}$ with an order relation $\leq$ such that $A + C \leq B + C$ and $\lambda A \leq \lambda B$ holds for all $A, B \in \mathbb{F}$ satisfying $A \leq B$ and for all $C \in \mathbb{F}$ and $\lambda \in \mathbb{R}_+$. If $\mathbb{F}$ is an ordered cone, then the set $\mathbb{F}_+ := \{ A \in \mathbb{F} \mid Z \leq A \}$ is said to be the *positive cone* of $\mathbb{F}$.
An ordered cone $\mathbb{F}$ has the *order cancellation property* if $A \leq B$ holds for all $A, B \in \mathbb{F}$ satisfying $A + C \leq B + C$ for some $C \in \mathbb{F}$, it is *Archimedean* if $A \leq B$ holds for all $A, B \in \mathbb{F}$ satisfying $nA + D \leq nB + C$ for some $C, D \in \mathbb{F}$ and all $n \in \mathbb{N}$, and it has the *Hukuhara property* if for all $A, B \in \mathbb{F}$ satisfying $A \leq B$ there exists some $D \in \mathbb{F}_+$ satisfying $A + D = B$.
Each Archimedean ordered cone has the order cancellation property, and each ordered cone having the cancellation property and the Hukuhara property has the order cancellation property.

Let $\mathbb{F}$ be an ordered cone having the order cancellation property.

On $\mathbb{G}$, define an order relation $\leq$ by letting

$$\langle A,B\rangle \leq \langle C,D\rangle$$

for all $\langle A,B\rangle, \langle C,D\rangle \in \mathbb{G}$ satisfying $A + D \leq B + C$.

<u>2.2. Theorem.</u>

Suppose $\mathbb{F}$ is an ordered cone having the order cancellation property. Then $\mathbb{G}$ is an ordered vector space, and j is isotone and inverse-isotone.

Moreover, $\mathbb{G}$ is Archimedean if and only if $\mathbb{F}$ is Archimedean, and $\mathbb{G}_+ = j(\mathbb{F}_+)$ holds if and only if $\mathbb{F}$ has the Hukuhara property.

A <u>semilattice cone</u> (or <u>upper semilattice cone</u> [13,14]) is an ordered cone $\mathbb{F}$ such that $A \vee B := \sup\{A,B\}$ exists for all $A, B \in \mathbb{F}$ and the identity $(A+C)\vee(B+C) = A\vee B + C$ holds for all $A, B, C \in \mathbb{F}$.

Each semilattice cone having the cancellation property has the order cancellation property, a semilattice cone $\mathbb{F}$ is Archimedean if and only if $A = B$ holds for all $A, B \in \mathbb{F}$ satisfying $B \leq A$ and $nA + D \leq nB + C$ for some $C, D \in \mathbb{F}$ and all $n \in \mathbb{N}$, and a semilattice cone $\mathbb{F}$ having the cancellation property and the Hukuhara property is Archimedean if and only if $A = Z$ holds for all $A \in \mathbb{F}_+$ satisfying $nA \leq C$ for some $C \in \mathbb{F}$ and all $n \in \mathbb{N}$.

A semilattice cone $\mathbb{F}$ has the <u>Riesz property</u> if the identity $A + B = A\vee B + A\wedge B$ holds for all $A, B \in \mathbb{F}$ for which $A\wedge B := \inf\{A,B\}$ exists. If $\mathbb{F}$ is a semilattice cone having the cancellation property and the Riesz property, then the identity $(A+C)\wedge(B+C) = A\wedge B + C$ holds for all $A, B, C \in \mathbb{F}$ for which $(A+C)\wedge(B+C)$ and $A\wedge B$ exist.

<u>2.3. Theorem.</u>

Suppose $\mathbb{F}$ is a semilattice cone having the cancellation property. Then $\mathbb{G}$ is a vector lattice, and j preserves finite suprema.

In particular, the identities $\langle A,B\rangle \vee \langle C,D\rangle = \langle (A+D)\vee(B+C), B+D\rangle$ and $\langle A,B\rangle \wedge \langle C,D\rangle = \langle A+C, (A+D)\vee(B+C)\rangle$ hold for all $\langle A,B\rangle, \langle C,D\rangle \in \mathbb{G}$.

Moreover, j preserves finite infima if and only if $\mathbb{F}$ has the Riesz property.

An ordered cone $\mathbb{F}$ is <u>(countably)</u> <u>order complete</u> if $\sup\{A_\gamma \mid \gamma \in \Gamma\}$ exists for each (countable) set $\{A_\gamma \in \mathbb{F} \mid \gamma \in \Gamma\}$ satisfying $A_\gamma \leq C$ for some $C \in \mathbb{F}$ and all $\gamma \in \Gamma$.

Each countably order complete semilattice cone having the cancellation property and the Hukuhara property is Archimedean.

2.4. Theorem.

Suppose $\mathbb{F}$ is a semilattice cone having the cancellation property and the Hukuhara property.
Then $\mathbb{G}$ is (countably) order complete if and only if $\mathbb{F}$ is (countably) order complete.

In the case where $\mathbb{F}$ is only an ordered semigroup, some of the previous results have been proven by Kaucher [4].

Normed Cones

If $\mathbb{F}$ is a cone and $d : \mathbb{F}\times\mathbb{F} \longrightarrow \mathbb{R}_+$ is a positively homogeneous translation-invariant metric, then

(i) $d(A,Z) = 0$ if and only if $A = Z$,

(ii) $d(A+B,Z) \leq d(A,Z) + d(B,Z)$, and

(iii) $d(\lambda A,Z) = \lambda d(A,Z)$

holds for all $A, B \in \mathbb{F}$ and $\lambda \in \mathbb{R}_+$, and

(iv) $d(A,Z) = d(B,Z)$

holds for all $A, B \in \mathbb{F}$ satisfying $A + B = Z$. These properties of the map $d(.,Z) : \mathbb{F} \longrightarrow \mathbb{R}_+$ suggest the following definition of a normed cone, but it should be noted that the existence of a map $\mathbb{F} \longrightarrow \mathbb{R}_+$ having properties (i) - (iv) is not equivalent to the existence of a positively homogeneous translation-invariant metric on $\mathbb{F}$; see also Mayer [7].

A normed cone is a cone $\mathbb{F}$ with a metric d which is translation-invariant and positively homogeneous. If $\mathbb{F}$ is a normed cone, then addition and scalar multiplication on $\mathbb{F}$ are continuous.
Each normed cone has the cancellation property.

Let $\mathbb{F}$ be a normed cone.
On $\mathbb{G}$, define a norm $\|.\| : \mathbb{G} \longrightarrow \mathbb{R}_+$ by letting

$$\| \langle A,B\rangle \| := d(A,B)$$

for all $\langle A,B\rangle \in \mathbb{G}$.

2.5. Theorem.

Suppose $\mathbb{F}$ is a normed cone.
Then $\mathbb{G}$ is a normed vector space, and j is isometric.

The previous result is due to Rådström [10]. For an extension of Theorem 2.5 to more general topological cones, see Urbanski [15].

Normed Ordered Cones

A normed ordered cone is an ordered cone $\mathbb{F}$ with a metric d which is translation-invariant and positively homogeneous and satisfies one (and thus all) of the following equivalent conditions:

- $d(A,B) \leq d(A,C)$ holds for all $A, B, C \in \mathbb{F}$ satisfying $A \leq B \leq C$;
- $d(B,C) \leq d(A,C)$ holds for all $A, B, C \in \mathbb{F}$ satisfying $A \leq B \leq C$;
- $d(A,B) \leq d(C,D)$ holds for all $A, B, C, D \in \mathbb{F}$ satisfying $B \leq A$, $D \leq C$, and $A + D \leq B + C$.

A metric satisfying these conditions is said to be chain isotone.

2.6. Theorem.

Suppose $\mathbb{F}$ is a normed ordered cone having the order cancellation property.
Then $\mathbb{G}$ is a normed ordered vector space.

A normed semilattice cone is a semilattice cone $\mathbb{F}$ with a metric d which is translation-invariant and positively homogeneous and satisfies one (and thus all) of the following equivalent conditions:

- d is chain isotone and $d((A+A)\vee(B+B), A+B) = d(A,B)$ holds for all $A, B \in \mathbb{F}$;
- $d(A,B) \leq d(C,D)$ holds for all $A, B, C, D \in \mathbb{F}$ satisfying $(A+A)\vee(B+B) + C + D \leq A + B + (C+C)\vee(D+D)$.

Each normed semilattice cone is Archimedean; in particular, each normed semilattice cone has the order cancellation property.

An M-normed semilattice cone (with unit) is a normed semilattice cone $\mathbb{F}$ such that

- $d((A+D)\vee(B+C), B+D) = \max \{ d(A,B) , d(C,D) \}$ holds for all $A, B, C, D \in \mathbb{F}$ satisfying $B \leq A$ and $D \leq C$ (and there exists some $E \in \mathbb{F}$ satisfying $d(E,Z) = 1$ and $A \leq B + E$ for all $A, B \in \mathbb{F}$ satisfying $d(A,B) \leq 1$).

2.7. Theorem.

Suppose $\mathbb{F}$ is a normed semilattice cone.
Then $\mathbb{G}$ is a normed vector lattice.
Moreover, $\mathbb{G}$ is an M-normed vector lattice (with unit) if and only if $\mathbb{F}$ is an M-normed semilattice cone (with unit).

The previous results can be extended to more general topological ordered cones, as indicated in [13].

3. CONES OF CONVEX SETS OCCURING IN INTERVAL MATHEMATICS

In this section, we study the cone properties of the classes of all norm balls of a normed vector space, all hypernorm balls of a hypernormed vector space, and all order intervals of a (normed) vector lattice. The cone properties of these classes of convex sets are studied with respect to the Minkowski addition of sets $+$ which is defined by letting

$$A + B \;:=\; \{\, a+b \mid a \in A\,,\ b \in B \,\}$$

for all nonempty subsets A and B of a (real) vector space $\mathbb{E}$, the usual multiplication of sets by positive scalars which is defined by letting

$$\lambda A \;:=\; \{\, \lambda a \mid a \in A \,\}$$

for all nonempty subsets A of $\mathbb{E}$ and (positive) scalars $\lambda \in \mathbb{R}_+$, and the inclusion of sets $\subseteq$. The cone properties of the class of all norm balls of a normed vector space $\mathbb{E}$ are also studied with respect to the Hausdorff distance Δ which is defined by letting

$$\Delta(A,B) \;:=\; \max\{\, \sup_A \inf_B \|a-b\|\,,\ \sup_B \inf_A \|b-a\| \,\}$$

for all nonempty bounded subsets A and B of $\mathbb{E}$, and those of the class of all order intervals of a normed vector lattice $\mathbb{E}$ are also studied with respect to the Moore distance δ which is defined by letting

$$\delta([a,b],[c,d]) \;:=\; \max\{\, \|a-c\|\,,\ \|b-d\| \,\}$$

for all order intervals $[a,b]$ and $[c,d]$ of $\mathbb{E}$ and which in certain cases agrees with the Hausdorff distance Δ. The proofs of the cone properties of these classes of convex sets cannot be included in these notes, but parts of them may be found in the paper of Fischer [1] and in [13] and [14].

Norm Balls

If $\mathbb{E}$ is a normed vector space with norm $\|.\|$, let

$$\bar{U} \;:=\; \{\, x \in \mathbb{E} \mid \|x\| \le 1 \,\}$$

denote the closed unit ball of $\mathbb{E}$. It is known and can be proven by a slight modification of the proof of [13; Theorem 6.2] that the identity

$$\Delta(A,B) \;=\; \inf\{\, \varepsilon \in (0,\infty) \mid A \subseteq B + \varepsilon\bar{U}\,,\ B \subseteq A + \varepsilon\bar{U} \,\}$$

holds for all nonempty bounded subsets A and B of $\mathbb{E}$.

Let $\mathbb{E}$ be a normed vector space with norm $\|.\|$. A subset A of $\mathbb{E}$ is a norm ball of $\mathbb{E}$ if there exist $m \in \mathbb{E}$ and $\mu \in \mathbb{R}_+$ satisfying

$$\langle m;\mu\rangle \;:=\; \{\, x \in \mathbb{E} \mid \|x-m\| \le \mu \,\} \;=\; m + \mu\bar{U} \;=\; A\,.$$

This midpoint-radius representation of a norm ball is clearly unique. Let $\mathbb{F}_b(\mathbb{E}, \|.\|)$ denote the class of all norm balls of $\mathbb{E}$, endowed with the Minkowski addition of sets, the usual multiplication of sets by positive scalars, and the inclusion of sets.

3.1. Theorem.

Suppose $\mathbb{E}$ is a normed vector space.
Then $\mathbb{F}_b(\mathbb{E}, \|.\|)$ is an Archimedean ordered cone having the Hukuhara property.
In particular, the identity $<m_1;\mu_1> + <m_2;\mu_2> = <m_1+m_2;\mu_1+\mu_2>$ holds for all norm balls $<m_1;\mu_1>$ and $<m_2;\mu_2>$ of $\mathbb{E}$.

We remark, however, that $\mathbb{F}_b(\mathbb{E}, \|.\|)$ may fail to be a semilattice cone; see [14; Example 5.4].

The following lemma can be used to give a simple proof of the properties of the Hausdorff distance on the class of all norm balls of $\mathbb{E}$:

3.2. Lemma.

Suppose $\mathbb{E}$ is a normed vector space.
Then the identity

$$\Delta(<m_1;\mu_1>, <m_2;\mu_2>) = \| m_1 - m_2 \| + | \mu_1 - \mu_2 |$$

holds for all norm balls $<m_1;\mu_1>$ and $<m_2;\mu_2>$ of $\mathbb{E}$.

Let $\mathbb{F}_b^\Delta(\mathbb{E}, \|.\|)$ denote the ordered cone $\mathbb{F}_b(\mathbb{E}, \|.\|)$ endowed with the Hausdorff distance Δ .

3.3. Theorem.

Suppose $\mathbb{E}$ is a normed vector space.
Then $\mathbb{F}_b^\Delta(\mathbb{E}, \|.\|)$ is an Archimedean normed ordered cone having the Hukuhara property.

This result can be proven by using Lemma 3.2 or [13; Lemma 3.2].

H y p e r n o r m B a l l s

If $\mathbb{E}$ is a vector space and $\mathbb{P}$ is an ordered vector space, then a map $h : \mathbb{E} \longrightarrow \mathbb{P}_+$ is a $\mathbb{P}$-hypernorm on $\mathbb{E}$ or briefly a hypernorm [1] if

(i) $h(x) = 0$ if and only if $x = 0$,

(ii) $h(x+y) \leq h(x) + h(y)$, and

(iii) $h(\alpha x) = |\alpha| h(x)$

holds for all $x, y \in \mathbb{E}$ and $\alpha \in \mathbb{R}_+$.
A $\mathbb{P}$-hypernorm h on a vector space $\mathbb{E}$ is splittable if for all $x \in \mathbb{E}$ satisfying $h(x) \leq p + q$ for some $p, q \in \mathbb{P}_+$ there exist $y, z \in \mathbb{E}$ satisfying $x = y + z$, $h(y) \leq p$, and $h(z) \leq q$, and it is surjective

if for all $p \in \mathbb{P}_+$ there exists some $x \in \mathbb{E}$ satisfying $h(x) = p$.

A <u>$\mathbb{P}$-hypernormed vector space</u> or briefly a <u>hypernormed vector space</u> is a vector space $\mathbb{E}$ with a $\mathbb{P}$-hypernorm h and will be denoted by $(\mathbb{E}, h)$. For example, if $\mathbb{E}$ is a normed vector space with norm $\|.\|$, then $(\mathbb{E}, \|.\|)$ is an $\mathbb{R}$-hypernormed vector space, and if $\mathbb{E}$ is a vector lattice with modulus $|.|$, then $(\mathbb{E}, |.|)$ is an $\mathbb{E}$-hypernormed vector space, and in either case the hypernorm is splittable and surjective.

Let $(\mathbb{E}, h)$ be a $\mathbb{P}$-hypernormed vector space. A subset A of $\mathbb{E}$ is a <u>hypernorm ball</u> of $\mathbb{E}$ if there exist $m \in \mathbb{E}$ and $p \in \mathbb{P}_+$ satisfying

$$<m;p> \;:=\; \{ x \in \mathbb{E} \mid h(x-m) \leq p \} \;=\; A \;.$$

Different from the case of norm balls, however, this <u>midpoint-radius representation</u> of a hypernorm ball need not be unique, as pointed out by Fischer [1]. Let $\mathbb{F}_b(\mathbb{E}, h)$ denote the class of all hypernorm balls of $\mathbb{E}$, endowed with the Minkowski addition of sets, the usual multiplication of sets by positive scalars, and the inclusion of sets.

<u>3.4. Theorem.</u>
Suppose $(\mathbb{E}, h)$ is a $\mathbb{P}$-hypernormed vector space such that $\mathbb{P}$ is Archimedean and h is splittable.
Then $\mathbb{F}_b(\mathbb{E}, h)$ is an Archimedean ordered cone.
In particular, the identity $<m_1;p_1> + <m_2;p_2> = <m_1+m_2;p_1+p_2>$ holds for all hypernorm balls $<m_1;p_1>$ and $<m_2;p_2>$ of $(\mathbb{E}, h)$.

The previous result improves [14; Theorem 5.3] where a weaker definition of an Archimedean ordered cone has been used; see also Fischer [1]. Under an additional assumption on the hypernorm, Theorem 3.4 can be improved as to yield the following complete extension of Theorem 3.1:

<u>3.5. Theorem.</u>
Suppose $(\mathbb{E}, h)$ is a $\mathbb{P}$-hypernormed vector space such that $\mathbb{P}$ is Archimedean and h is splittable and surjective.
Then $\mathbb{F}_b(\mathbb{E}, h)$ is an Archimedean ordered cone having the Hukuhara property.

Since hypernormed vector spaces generalize normed vector spaces, $\mathbb{F}_b(\mathbb{E}, h)$ may fail to be a semilattice cone, by the remark following Theorem 3.1.

It would be interesting to know whether Theorem 3.3 can be extended to the hypernormed case by replacing the norm in the definition of the Hausdorff distance by the hypernorm h (in the case where $\mathbb{P}$ is order complete).

In [14], the corresponding question for a different class of convex sets in a hypernormed vector space has been answered in the negative.

Order Intervals

Let $\mathbb{E}$ be a vector lattice with modulus $|.|$. A subset A of $\mathbb{E}$ is an order interval of $\mathbb{E}$ if there exist $a, b \in \mathbb{E}$ satisfying $a \leq b$ and

$$[a,b] := \{ x \in \mathbb{E} \mid a \leq x \leq b \} = A .$$

This lower-bound-upper-bound representation of an order interval is unique and yields a (unique) midpoint-radius representation of an order interval with respect to the modulus $|.|$; see e.g. [14; Proposition 6.1]. Let $\mathbb{F}_b(\mathbb{E}, |.|)$ denote the class of all order intervals of $\mathbb{E}$, endowed with the Minkowski addition of sets, the usual multiplication of sets by positive scalars, and the inclusion of sets.

3.6. Theorem.

Suppose $\mathbb{E}$ is a vector lattice.
Then $\mathbb{F}_b(\mathbb{E}, |.|)$ is a semilattice cone having the cancellation property, the Hukuhara property, and the Riesz property.
In particular, the identities $[a,b] + [c,d] = [a+c,b+d]$ and $[a,b] \vee [c,d] = [a\wedge c,b\vee d]$ hold for all order intervals $[a,b]$ and $[c,d]$ of $\mathbb{E}$, and the identity $[a,b] \wedge [c,d] = [a\vee c,b\wedge d]$ holds for all order intervals $[a,b]$ and $[c,d]$ of $\mathbb{E}$ having nonempty intersection.
Moreover, $\mathbb{F}_b(\mathbb{E}, |.|)$ is Archimedean if and only if $\mathbb{E}$ is Archimedean, and $\mathbb{F}_b(\mathbb{E}, |.|)$ is (countably) order complete if and only if $\mathbb{E}$ is (countably) order complete.

The previous result has been proven in [14; Lemma 6.2, Lemma 6.3, Lemma 6.5, and Theorem 6.6]. In the case where $\mathbb{E}$ is Archimedean, some of the assertions of Theorem 3.6 can be obtained from Theorem 3.5, but even in that case Theorem 3.6 provides more information on $\mathbb{F}_b(\mathbb{E}, |.|)$ than Theorem 3.5 does. Therefore, it seems not to be convenient to consider the order intervals of $\mathbb{E}$ as hypernorm balls of $(\mathbb{E}, |.|)$.

Let now $\mathbb{E}$ be a normed vector lattice with modulus $|.|$ and norm $\|.\|$, and let $\mathbb{F}_b^{\delta}(\mathbb{E}, |.|)$ denote the ordered cone $\mathbb{F}_b(\mathbb{E}, |.|)$ endowed with the Moore distance δ . The Moore distance has been introduced by Moore [8] in the case $\mathbb{E} = \mathbb{R}$, and it has been used by Jahn [3] in the general case.

3.7. Theorem.

Suppose $\mathbb{E}$ is a normed vector lattice.
Then $\mathbb{F}_b^{\delta}(\mathbb{E}, |.|)$ is a normed semilattice cone.

Moreover, $\mathbb{F}_b^{\delta}(\mathbb{E}, |.|)$ is an M-normed semilattice cone (with unit) if and only if $\mathbb{E}$ is an M-normed vector lattice (with unit).

In the general case, the Moore distance δ may differ from the Hausdorff distance Δ, as can be seen from [14; Example 6.12]. However, we have the following result which can also be used to give a simple proof of the properties of the Hausdorff distance on the class of all order intervals of an M-normed vector lattice with unit:

3.8. Lemma.

Suppose $\mathbb{E}$ is an M-normed vector lattice with unit.
Then the identity

$$\Delta([a,b],[c,d]) = \delta([a,b],[c,d])$$

holds for all order intervals $[a,b]$ and $[c,d]$ of $\mathbb{E}$.

For a proof of Lemma 3.8, see [14; Lemma 6.11].

Let $\mathbb{F}_b^{\Delta}(\mathbb{E}, |.|)$ denote the ordered cone $\mathbb{F}(\mathbb{E}, |.|)$ endowed with the Hausdorff distance Δ.

3.9. Theorem.

Suppose $\mathbb{E}$ is a normed vector lattice.
Then $\mathbb{F}_b^{\Delta}(\mathbb{E}, |.|)$ is a normed ordered cone having the order cancellation property.
Moreover, $\mathbb{F}_b^{\Delta}(\mathbb{E}, |.|)$ is an M-normed semilattice cone with unit if and only if $\mathbb{E}$ is an M-normed vector lattice with unit.

For a proof of Theorem 3.9, see [14; Theorem 6.13 and Theorem 6.14].
We remark that in the case where $\mathbb{E}$ is an arbitrary normed vector lattice, $\mathbb{F}_b^{\Delta}(\mathbb{E}, |.|)$ may fail to be a normed semilattice cone since the Hausdorff distance need not be compatible with the semilattice structure of $\mathbb{F}_b(\mathbb{E}, |.|)$, as can be seen from [14; Example 6.10].

In the case where $\mathbb{E}$ is an arbitrary normed vector lattice of dimension greater than one, only the one-point sets of $\mathbb{E}$ are at the same time norm balls and order intervals of $\mathbb{E}$. However, in the case where $\mathbb{E}$ is even an M-normed vector lattice with unit, each norm ball of $\mathbb{E}$ is an order interval of $\mathbb{E}$ since there exists some $e \in \mathbb{E}_+$ satisfying $\overline{U} = [-e,e]$. In this case, $\mathbb{F}_b(\mathbb{E}, |.|)$ may be considered as the semilattice completion of $\mathbb{F}_b(\mathbb{E}, \|.\|)$, and the identity $\overline{U} = [-e,e]$ together with the role of the closed unit ball $\overline{U}$ in one of the equivalent definitions of the Hausdorff distance may also serve as an intuitive explanation of the compatibility of the Hausdorff distance with the semilattice structure of $\mathbb{F}_b(\mathbb{E}, |.|)$.

4. REMARKS

As an abstraction of the structure of the class of all order intervals of an ordered vector space, Mayer [7] introduced the notion of a quasilinear space. Mayer also considered norms and metrics on a quasilinear space, but it appears that no ordered quasilinear spaces have been studied in the literature. This is somewhat surprising since ordered quasilinear spaces would reflect the inclusion of order intervals and would thus allow for a formulation of the subdistributive law for order intervals without any restriction on the scalars. On the other hand, with regard to the restricted distributive law in quasilinear spaces and the fact that a quasilinear space cannot be embedded into a vector space such that the embedding map is additive and homogeneous (and not only positively homogeneous), as pointed out by Kracht and Schröder [5], it seems to be convenient to generalize one step further and to restrict multiplication by scalars to positive scalars alone. This leads from quasilinear spaces to cones.

While the Minkowski addition of sets, the usual multiplication of sets by positive scalars, and the distances of Hausdorff and Moore are clearly related to the structure of the underlying vector space, this is not the case for the inclusion of sets. For order intervals of an ordered vector space $\mathbb{E}$, a different order relation $\prec$ can be defined by letting $[a,b] \prec [c,d]$ if $a \le c$ and $b \le d$ holds; see Nickel [9] and Jahn [3]. The order relation $\prec$ extends the order relation on $\mathbb{E}$, and if $\mathbb{E}$ is even a vector lattice, then the order intervals of $\mathbb{E}$ form a lattice with respect to the order relation $\prec$, as pointed out by Nickel [9]. Let $\mathbb{F}_b(\mathbb{E}, \prec)$ denote the class of all order intervals of $\mathbb{E}$, endowed with the Minkowski addition of sets, the usual multiplication of sets by positive scalars, and the order relation $\prec$. Then $\mathbb{F}_b(\mathbb{E}, \prec)$ is a lattice cone having the order cancellation property and the Riesz property but lacking the Hukuhara property. Furthermore, if $\mathbb{E}$ is a normed vector lattice, let $\mathbb{F}_b^{\delta}(\mathbb{E}, \prec)$ denote the ordered cone $\mathbb{F}_b(\mathbb{E}, \prec)$ endowed with the Moore distance δ . Then $\mathbb{F}_b^{\delta}(\mathbb{E}, \prec)$ is a normed lattice cone. Since the cone properties of $\mathbb{F}_b(\mathbb{E}, \prec)$ and $\mathbb{F}_b^{\delta}(\mathbb{E}, \prec)$ are slightly different from those of $\mathbb{F}_b(\mathbb{E}, |.|)$ and $\mathbb{F}_b^{\delta}(\mathbb{E}, |.|)$, they lead to different embedding theorems for order intervals reflecting the different properties of the order relation $\subseteq$ and $\prec$.

For the class of all order intervals of an (M-normed) vector lattice (with unit) $\mathbb{E}$, it is also possible to prove embedding theorems which are concrete in the sense that the embedding vector lattice is defined to be the Cartesian product $\mathbb{E}\times\mathbb{E}$, endowed with the componentwise defined addition, scalar multiplication, and order relation (and the sup-norm), and that the embedding map is defined by using the lower-bound-upper-bound representation of order intervals. These concrete embedding theorems can be obtained from an (isometrically) isomorphic representation of the (abstract) embedding vector lattice by $\mathbb{E}\times\mathbb{E}$, and the different properties of the order relations $\subseteq$ and $\prec$ are then reflected by the different properties of the embedding map. Such concrete embedding theorems for $\mathbb{F}_b(\mathbb{E}, |.|)$ and $\mathbb{F}_b^{\delta}(\mathbb{E}, |.|)$ have been proven in [14]; see also [13] for the case $\mathbb{E} = \mathbb{R}$.

Apart from the embedding theorems which can be obtained from the results of Sections 2 and 3, it appears that the investigation of the cone properties of classes of convex sets occuring in interval mathematics, which is motivated by Rådström's method for proving embedding theorems, may also be interesting for its own sake since it seems to be helpful for understanding the different properties of norm balls, hypernorm balls, and order intervals, and those of different order relations and metrics.

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