INTERVAL TEST AND EXISTENCE THEOREM

Shen Zuhe
Nanjing University
Nanjing
The People's Republic of China

1. Introduction

Interval methods have been introduced for computationally verifiable sufficient conditions for existence, uniqueness and convergence, for solving finite dimensional nonlinear systems [2], [7], [8], [9], [13] and for nonlinear operator equations in infinite dimensional spaces [11]. The methods can also be used to discuss some classical existence theorems [20], [21]. The conditions, like bounded inverse of the derivative, norm coercivity, or uniform monotonicity guarantee the homeomorphism of a differentiable function, and also, the nonsingularity assumption of the partial derivative guarantee the existence of a implicit function [14], [19]. In this paper, using interval methods, precisely, the centred form of the Newton-transform of f [3] and the Moore-like test for the Krawczyk operator [2], [8], we will derive some computationally verifiable sufficient conditions for a function f to be a homeomorphism and the global implicit function theorem. The cases of the function f(x) or f(x,y) to be local Lipschitz continuous and continuously differentiable are of special interest. Some further results for these classes of functions and a sufficient condition for the feasibility of the numerical continuation method are given.

2. Interval test

The Moore test [8] for a nonlinear system

$$f(x) = 0 \tag{1}$$

is based on the Krawczyk operator [2]

$$K(X) = y - \Gamma f(y) + \{I - \Gamma F'(X)\}(X - y).$$

The operator consists of the sum of the point iteration $\varphi(y) = y - \Gamma f(y)$ and a symmetric interval vector $\{I - \Gamma F'(X)\}(X - y)$. One can choose the real matrix Γ for the Moore test. It is also possible to change the second part of the Krawczyk operator such that the Moore test becomes (at least sometimes!) more efficient. If we recognize that the Krawczyk operator is essentially the mean value form extension of $\varphi(y)$, then instead, other form extensions may also be used. This is indeed worth having and the centred form extension is perhaps a good choice and of practical value. The following example from [10] may illustrate the problem. Let

$$f(x) = \begin{pmatrix} x_1^2 + 0.25x_2 - 0,1725 \\ x_2^2 - 3x_1 + 0.46 \end{pmatrix}$$

$$X = \begin{pmatrix} [0,\ 0.5] \\ [0,\ 1] \end{pmatrix},\ y = m(X) = \begin{pmatrix} 0.25 \\ 0.5 \end{pmatrix},\ \Gamma = \begin{pmatrix} 0.8 & -0.2 \\ 2.4 & 0.4 \end{pmatrix}$$

we have [10]

$$K(X) = \begin{pmatrix} [0.03,\ 0.43] \\ [-0.02,\ 0.98] \end{pmatrix} \not\subset X.$$

By contrast, the centred form extension of $\varphi(x) = x - \Gamma f(x)$ on X is

$$K(X) = \begin{pmatrix} [0.1\ ,\ 0.33] \\ [0.23,\ 0.73] \end{pmatrix}.$$

It is clear that $K_c(X) \subset K(X)$ and $K_c(X) \subset X$. If the Moore test is true for $K_c(X)$ instead $K(X)$, it is then at least better to use $K_c(X)$ for this example.

The centred form extension was first suggested by Moore [7]. Explicit formulas for these forms were formed by Hansen [5] for polynomials and by Ratschek [18] for rational functions. Recursively defined centred forms were recently introduced by Krawczyk [4].

Let $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ be a symmetric interval, for an interval vector $X = (X_1, X_2, \cdots . X_n)$ and an interval matrix $A = (A_{ij})$, define [12]

$$\|X\|_\omega = \min\{\alpha \geqq 0 \mid X \subset \alpha\omega\}$$

$$\|A\|_\omega = \min\{\alpha \geqq 0 \mid A\omega \subset \alpha\omega\}$$

$$sp_\omega(X) = \|X - m(X)\|_\omega$$

and call them the interval norm, the interval matrix norm and the width of interval X respectively. It is easy to check the norm properties of a norm.

Suppose that $X \subset D \subset R^n$ is an interval, $f\colon D \subset R^n \to R^n$ is continuous on X and satisfies

$$f(x) - f(y) = h(x,y)(x-y), \quad x \in X, \ y \in R^n, \tag{2}$$

where $h(x,y) = (h_{ij}(x,y))$ and $h_{ij}(x,y)\colon X \times R^n \to R^1$, $i,j = 1,2,\cdots,n$. For $n > 1$, there are in general infinitely many such matrices. Suppose that $h_{ij}(x,y)$ has an inclusion monotonic interval extension $H_{ij}(X,y)$, let

$$H(X,y) = (H_{ij}(X,y)), \tag{3}$$

and the Lipschitz condition

$$d(H(X,y), h(x,y)) \leqq \lambda d(X,y), \quad x \in X,$$

be satisfied. One possible such Lipschitz constant λ is

$$\lambda = \frac{2\, sp_\omega(H(X,y))}{sp_\omega(X)}, \quad \text{for} \quad sp_\omega(X) > 0.$$

Define

$$F(X,y) = f(x) + H(X,y)(X - y), \quad y \in R^n$$

call it the centred form extension of f on X. For brevity, write $F(X) = F(X, m(X))$.

The following two special cases are of interest

(a) Suppose that $f(x): D \subset R^n \to R^n$ is local Lipschitz continuous on D [16]. For such maps, one can assign to each X, a certain collection $\partial f(x)$ of linear transformation from R^n into R^n, called the generalized derivative and

$$\partial f(x) = \bigcap_{r>0} \overline{CO}\{f'(z) \mid f'(z) \text{ exists and } \|z - x\|_\infty < r\},$$

where $\overline{CO}$ is short for the closure of the convex hull.
Given $x_1, x_2 \in R^n$, if $[x_1, x_2] \subset D$, then we have the following Lipschitz mean value theorem

$$f(x_1) - f(x_2) = A(x_1 - x_2), \quad A \in \overline{CO}_{z \in [x_1, x_2]} \{\partial f(z)\}.$$

For each interval $X \subset D$, let

$$\partial f(X) = ([\underline{A_{ij}}, \overline{A_{ij}}]), \tag{4}$$

where

$$\underline{A_{ij}} = \min\{A_{ij} \mid A \in \overline{CO} \bigcup_{x \in X} \partial f(x)\},$$

$$\overline{A_{ij}} = \max\{A_{ij} \mid A \in \overline{CO} \bigcup_{x \in X} \partial f(x)\};$$

then $\partial f(X)$ is an inclusion monotonic interval extension of $\partial f(x)$ on X and

$$F(X) = f(m(X)) + \partial f(X)(X - m(X))$$

is a centred form extension of $f(x)$ on X.

(b) Let $f(x) = (f_1(x), f_2(x), \cdots, f_n(x)): D \subset R^n \to R^n$ with $f_i(x) \in C^1(D)$, then

$$f(x) - f(z) = f'(\xi(x,z))(x - z), \quad x, z \in D,$$

i.e., $h(x,z) = f'(\xi(x,z))$ can be defined. Let $F'(X)$ be an inclusion monotonic interval extension of $f'(x)$ on X, then

$$F(X) = f(z) + F'(X)(X - z)$$

is a centred form extension of f on X.

Krawczyk [3] also investigated the centred form extension of the Newton-transform and the corresponding interval operator for solving the equation (1), where $f\colon D \subset R^n \to R^n$ is continuous in the open set D. A simple way to convert the equation (1) to be solved into a fixed point problem is by introduction of the Newton-transform of f defined by

$$p(x) = x - \Gamma f(x),$$

where Γ is a nonsingular real matrix. Let

$$r(x,y) = I - \Gamma h(x,y)$$

and

$$R(X,y) = I - \Gamma H(X,y),$$

where h(x,y) and H(X,y) are given in (2) and (3), respectively, then

$$P(X) = p(m(X)) + R(X)(X - m(X))$$

is the centred form extension of the Newton-transform p(x).

<u>Theorem 1</u> [3].

(a) If $P(X) \subseteq X$, then there exists a fixed point $\hat{x} \in X$ of p(x);

(b) If $P(X) \subseteq \mathrm{Int}(X)$, then $\|R(X)\|_\omega \leqq \lambda < 1$.

<u>3. The inverse function theorems (homeomorphism)</u>.

The inverse function is one of the most important notions in nonlinear analysis. The main theorem, for instance, is in the following ([14], Theorem 5.2.1, p.125):

Theorem 2.

Suppose that $f\colon D \subset R^n \to R^n$ has an F-derivative on a neighborhood of x_0, which is continuous at x_0 and that $f'(x_0)$ is nonsingular, then f is a local homeomorphism at x_0.

In this section, we will make use of interval methods to discuss the problem under weaker restrictions. In general, no differentiability of f on D is required.

Suppose that $f\colon D \subset R^n \to R^n$ is continuous on X and satisfies

$$f(x) - f(y) = h(x,y)(x - y),$$

where h(x,y) is given in (2). Consider the Newton-transform of f

$$p(x) = x - \Gamma(f(x) - f(x_0))$$

and its centred form extension

$$P(X) = x_0 + (I - \Gamma H(X))(X - x_0), \tag{5}$$

where H(X) is given in (3). We have

Theorem 3.

If for $x_0 \in X$, there exists an interval $X = [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$ such that

$$P(X) \subseteq \mathrm{Int}(X),$$

then f(x) is a local homeomorphism at x_0.

Proof. The only thing we need to prove is the nonsingularity of H(x) at x_0 (cf. [14], Theorem 5.1.9, p. 124). Since X is an interval, by Theorem 1, we have

$$\| I - \Gamma H(x) \| \leqq \lambda < 1.$$

For brevitiy, we write $\| \ \|$ instead of $\| \ \|_\omega$ in the sequel. This proves that, for each $x \in X$, and of course, for x_0, H(x) is nonsingular and so is $H(x_0)$.

□

Specially, suppose that $f: D \subset R^n \to R^n$ is local Lipschitz continuous on D. For $X \subset D$, define

$$H(X) = \partial f(X), \tag{6}$$

where $\partial f(X)$ is defined in (4), then we have

$$p(x) \in P(X)$$

and Theorem 3 is true for such functions.

Moreover, if f is continuously differentiable at x_0, define

$$H(X) = [\min_{x \in X} f'(x), \max_{x \in X} f'(x)],$$

then we have

$$p(x) \in P(X)$$

and the following stronger conclusion [20].

Theorem 4.

$f'(x)$ is nonsingular at x_0 if and only if there exists an interval $X = [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}] \subset D$ such that

$$P(X) \subseteq \mathrm{Int}(X). \tag{7}$$

Proof. We have only to prove the necessity. Let $\Gamma = [f'(x_0)]^{-1}$, from (5), we have

$$P(X) = x_0 + (I - \Gamma H(X))(X - x_0),$$

therefore, for any $x \in P(X)$, we have

$$x = x_0 + z,$$

where $z \in (I - \Gamma H(X))(X - x_0)$. Since

$$\begin{aligned} \|I - \Gamma H(X)\| &\leq \|\Gamma\| \|f'(x_0) - F'(X)\| \\ &= \|\Gamma\| \|f'(x_0) - [\min_{x \in X} f'(x), \max_{x \in X} f'(x)]\| \end{aligned}$$

and f is continuous at x_0, there exists $\delta > 0$ such that, for $X = [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$, $\| I - \Gamma H(X) \| < 1$ and $\| (I - \Gamma H(X))(X - x_0) \| < \frac{\delta}{2}$, therefore

$$\| P(X) - x_0 \| < \frac{\delta}{2},$$

and (7) is proved.

□

Corollary 5.

If for $x_0 \in X^{(0)}$, there exists an interval $X = [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}] \subset X^{(0)} \subset D$ such that

$$P(X) \subseteq \mathrm{Int}(X),$$

then f(x) is a local homeomorphism at x_0.

4. The global inverse function theorem.

Following the above discussion, we will give a computationally verifiable sufficient condition for the global inverse function theorem. In this way, the theory of continuation is required.

Definition 1.

The continuous mapping $f: D \subset R^n \to R^n$ satisfies condition (L) if for every $(x_0, y) \in D \times R^n$ and every path $q: [0,b) \to D$, $b \leq 1$ such that $f(q(t)) = (1-t)f(x_0) + ty$, $0 \leq t < b$, there exists a sequence $t_n \to b$ as $n \to \infty$ such that $\lim_{n\to\infty} q(t_n)$ exists and is in D.

Theorem 6 (Plastock [15]).

Let $f: D \subset R^n \to R^n$ be a local homeomorphism at each point $x \in D$. The condition (L) is necessary and sufficient for f(x) to be a homeomorphism from D onto R^n.

In connection with interval methods, we have

Theorem 7.

If for each $x \in D$, there exist an interval $\overline{X}_x = [x - \frac{\delta(x)}{2},\ x + \frac{\delta(x)}{2}]$ and a nonsingular real matrix $\Gamma(x)$, continuous in x such that

(a) $\|\Gamma(x)\| \leqq M$, M = constant, $x \in D$;

(b) there exists a constant α, $0 \leqq \alpha < 1$ such that

$$|P(X_x) - m(X_x)| \leqq \alpha \frac{(x)}{2},$$

then $f(x)$ is a homeomorphism from D onto R^n.

Proof. By (b), $f(x)$ is a local homeomorphism at each point $x \in D$. The only problem remained to prove is that $f(x)$ satisfies condition (L). It follows that, for each $x \in X_x$, and consequently, for each $x \in D$, we have

$$\|I - \Gamma H(x)\| \leqq \alpha < 1,$$

therefore

$$\|H(x)^{-1}\| \leqq \frac{\|\Gamma(x)\|}{1-\alpha} \leqq \frac{M}{1-\alpha}.$$

Now, for every $(x_0, y) \in D \times R^n$ and every path $q(t)\colon [0,b) \to D$ such that

$$f(q(t)) = (1-t)f(x_0) + ty = p(t), \quad 0 \leqq t < b,$$

we have

$$f(q(t_1)) - f(q(t_2)) = p(t_1) - p(t_2)$$

$$= (t_1 - t_2)(y - f(x_0)), \quad t_1, t_2 \in [0,b),$$

and

$$F(q(t_1)) - F(q(t_2)) = H(q(t_1), q(t_2))(q(t_1) - q(t_2)).$$

It is clear that $H(q(t_1), q(t_2))$ is also nonsingular and

$$\|H(q(t_1), q(t_2))^{-1}\| \leqq \frac{M}{1-\alpha}, \text{ for any } t_1, t_2 \in [0,b),$$

therefore

$$\|q(t_1) - q(t_2)\| \leqq \frac{M}{1-\alpha} \|y - f(x_0)\| \, |t_1 - t_2| .$$

It means that q(t) is Lipschitz and $\lim_{t\to b} q(t)$ exists. This proves that f is a homeomorphism from D onto R^n. □

<u>Corollary 8</u>.

If f(x) is local Lipschitz continuous on D and define H(X) as in (6), then the conditions (a) and (b) in Theorem 7 are sufficient for f(x) to be a homeomorphism from D onto R^n.

The other special case is that f(x) is continuously differentiable on D. Let Ω be the set of all continuous mappings ω: $R_+ \to R_+$ satisfying the following properties

$$\omega(t) > 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad \int_1^\infty \frac{dt}{\omega(t)} = \infty.$$

<u>Theorem 9</u> (M. and S. Radulescu [17] and Mihai Cristea [6]).

Let $a,b \in R^n$, $a < b$ and $\sigma = \{x \in R^n,\ a \leqq x < b\}$, u,v: $\sigma \to R_+$, u be continuous and v integrable on σ. If $c \geqq 0$ and $\omega \in \Omega$ is a map with the properties

$$\int_a^b v(y)\omega(u(y))dy < \infty$$

$$u(x) \leqq c + \int_a^x v(y)\omega(u(y))dy , \quad x \in \sigma,$$

then the following inequality holds

$$\int_0^{u(x)} \frac{ds}{\omega(s)} \leqq \int_0^x v(y)dy.$$

With Theorem 9, we have

<u>Theorem 10</u>.

Let $\omega(x) \in \Omega$, f: $D \subset R^n \to R^n$ be a continuously differentiable map. If for each $x \in D$, there exist an interval X_x with the width $\delta(x) > 0$, $x \in X_x$ and a nonsingular real matrix $\Gamma(x)$ such that

(a) $\|\Gamma(x)\| \leqq \omega(\|x\|)$, $x \in D$;

(b) there exists a constant α, $0 \leqq \alpha < 1$ such that

$$|P(X_x) - m(X_x)| \leqq \alpha \frac{\delta(x)}{2},$$

then $f(x)$ is a homeomorphism from D onto R^n.

Proof. By the condition of the theorem, $f(x)$ is a local homeomorphism at each point x in D and $f'(x)$ is nonsingular with

$$\|f'(x)^{-1}\| \leqq \frac{1}{1-\alpha}\|\Gamma(x)\| \leqq \frac{\omega(\|x\|)}{1-\alpha}, \quad x \in D.$$

Now, for every $x_0, y \in R^n$, $p(t) = (1-t)f(x_0) + ty$, $0 \leqq t < 1$, $q(t): [0,b) \to R^n$, $b \leqq 1$ such that

$$q(0) = x_0, \quad f(q(t)) = p(t), \quad t \in [0,b), \tag{8}$$

we may assume that the path $q(t) \in C^1$. Differentiating the equality (8), we obtain

$$f'(q(t))q'(t) = p'(t),$$

whence $q'(t) = (f'(q(t)))^{-1}(y - f(x_0))$, $t \in [0,b)$. Since

$$\begin{aligned}\|q(t) - q(0)\| &\leqq \int_0^t \|q'(s)\|\,ds \leqq \int_0^t \|f'(q(s))^{-1}\|\,\|p'(s)\|\,ds \\ &\leqq \int_0^t \frac{\omega(\|q(s)\|)}{1-\alpha}\|p'(s)\|\,ds \\ &= \frac{\|f(x_0) - y\|}{1-\alpha}\int_0^t \omega(\|q(s)\|)\,ds, \\ &\qquad 0 \leqq t < b,\end{aligned}$$

we have

$$\|q(t)\| \leqq \|q(0)\| + \frac{\|f(x_0) - y\|}{1-\alpha}\int_0^t \omega(\|q(s)\|)\,ds, \qquad 0 \leqq t < b.$$

By applying Theorem 9 to the preceeding inequality, it follows that

$$\int_{\|q(0)\|}^{\|q(t)\|} \frac{ds}{\omega(s)} \leqq \int_0^t \frac{\|f(x_0) - y\|}{1-\alpha} ds ,$$

whence there exists $k > 0$ such that $\|q(t)\| \leqq k$, $0 \leqq t < b$. We see that $0 \leqq t_1, t_2 < b$ implies

$$\|q(t_1) - q(t_2)\| \leqq \int_{t_1}^{t_2} \|q'(s)\| ds \leqq \int_{t_1}^{t_2} \|f'(q(s))^{-1}\| \|f(x_0) - y\| ds$$
$$< \frac{\|f(x_0) - y\|}{1-\alpha} \omega(k) |t_1 - t_2| .$$

therefore q is Lipschitz. Consequently, it follows that $\lim_{t\to b} q(t)$ exists. Thus, we have proven that f satisfies condition (L). □

Note. The assumption "$\Gamma(x)$ is uniformly continuous on D or $\Gamma(x)$ is a constant nonsingular matrix" is sufficient for the condition (a) in Theorem 10.

5. The feasibility of the numerical continuation method.

Many iterative methods will converge to a solution x^* of $f(x) = 0$ only if the initial approximations are sufficiently close to x^k. The continuation methods may be considered as an attempt to widen this domain of convergence of a given method or as a procedure to obtain sufficiently close starting points. The method can be described as follows: Let $f: D \subset R^n \to R^n$ be a given mapping and consider the problem of solving the equations (1). This problem is imbedded into a family of problems of the form

$$G(t,x) = 0, \tag{9}$$

where $t \in [0,1]$ is a parameter. The imbedding is chosen so that at $t = 0$, the solution of (9) is a known point x_0, while at $t = 1$, the solution x_1 of (9) also solves (1). For example, $G(t,x)$ might have the form

$$G(t,x) = tf(x) + (1-t)f(x_0) . \tag{10}$$

Suppose now that there exists a continuous solution curve $x(t): [0,1] \to D$ of (9) starting at x_0, the continuation method then involves proceeding in some as yet unspecified manner along or near this curve $x = x(t)$ from the initial point $x_0 = x(0)$ to the final $x_1 = x(1)$. For each t, consider the iterative process

$$x^{n+1} = G(t,x^n).$$

In general, this process will converge to x(t) only for starting values near that point, and we cannot hope that we have convergence for the process with t = 1 starting from x(0). This lead to the following numerical continuation process: a partition of [0,1]

$$0 = t_0 < t_1 < \cdots < t_n = 1 \tag{11}$$

and a sequence of integers $\{j_k\}$, $k = 1,2,\cdots,N-1$ is chosen such that the points

$$\begin{aligned} x_k^{j+1} &= G(t_k,x_k^j), \quad j = 0,1,\cdots,j_{k-1}, \ k = 1,2,\cdots,N-1 \ , \\ x_{k+1}^0 &= x_k^{j_k} \ , \quad x_1^0 = x(0) \end{aligned} \tag{12}$$

are well-defined and such that

$$x_N^{j+1} = G(1, x_N^j), \quad j = 0,1,2,\cdots \tag{13}$$

converges to x(1) as $j \to \infty$.

Although many authors have discussed the numerical continuation approach, little attention appears to have been paid to its feasibility. Avila [1] established the feasibility of the numerical continuation process under nearly minimal conditions.

<u>Definition 2</u>.

If a partition (11) exists so that with some sequence of integers $\{j_n\}$ the entire process (12), (13) is well-defined and so that (13) converges to x(1), then the numerical continuation process (12), (13) is called feasible.

Theorem 11 (Avila [1]).

Let $G: [0,1]\times D \subset [0,1]\times R^n \to R^n$. Assume that $x: [0,1] \to D$ is continuous and satisfies $x(t) \equiv G(t,x(t))$.
Let G have a strong partial derivative with respect to x at $(t,x(t))$ for every $t \in [0,1]$. If the spectral radius of $G_x(t,x(t))$

$$\rho(G_x(t,x(t))) < 1$$

for all $t \in [0,1]$, then the numerical continuation process (12), (13) is feasible.

Two basic assumptions are involved in Avila's theorem: The existence of the solution curve $x(t)$ and the spectral radius $\rho(G_x(t,x(t)) < 1$ for all $t \in [0,1]$, which are certainly difficult to test. In this section, we will also make use of interval analysis to give simpler sufficient conditions for Avila's theorem.

Let D be an open set, $f: D \subset R^n \to R^n$ be continuously differentiable on D. If for each $x \in D$, there exists an interval X_x with the width $\delta(x)$, $x \in X_x$ and a nonsingular real matrix $\Gamma(x)$, continuous in x such that the conditions (a), (b) in Theorem 10 are satisfied, then by Theorem 10 f is homeomorphism from D onto R^n. That is, for each $t \in [0,1]$, there exists a unique point $x(t) \in D$ such that

$$G(t,\gamma(t)) = tf(x(t))+(1-t)f(x_0) = 0$$

and $x(t)$ is continuous on $[0,1]$. At the same time, for each $t \in [0,1]$ and for the interval mapping

$$P_{x(t)}(X) = m(X)-\Gamma(x(t))(tf(x(t))+(1-t)f(x_0))+(I-\Gamma(x(t))H(X))(X-m(X)),$$

there exists an interval $X_{x(t)}$ with the width $\delta(x(t))$ such that

$$|P_{x(t)}(X_{x(t)})-m(X_{x(t)})| \leqq \alpha \frac{\delta(x(t))}{2}$$

and consequently

$$\|I - \Gamma(x(t))H(X_{x(t)})\| \leqq \alpha < 1, \tag{14}$$

therefore, for each t, the sequence

$$x^{k+1}(t) = x^k(t) - \Gamma(x^k(t))G(t,x^k(t)) \tag{15}$$

converges to the unique solution x(t) for any starting point $x^0(t)$ in $X_{x(t)}$.

We now turn our attention to the Newton-transform (15) as the local process in numerical continuation method. Consider the mapping $p(t,x)\colon [0,1]\times D \to R^n$ given by

$$p(t,x) = x - \Gamma(x)G(t,x).$$

For a certain partition $\{t_k\}$, $k = 0,1,\cdots,N$ of $[0,1]$ and integers $\{j_k\}$ $k = 1,2,\cdots,N-1$, the process is defined by the equations

$$\begin{aligned} x_k^{j+1} &= x_k^j - \Gamma(x(t_j))G(t_j,x_k^j),\ k = 1,2,\cdots,N-1,\ j=0,1,\cdots,j_{k-1}, \\ x_{k+1}^0 &= x_k^{j_k},\quad x_1^0 = x(0) \end{aligned} \tag{16}$$

followed by

$$x_N^{j+1} = x_N^j - \Gamma(x(1))G(1,x_N^j),\quad x_N^0 = x_{N-1}^{j_{N-1}},\ j = 0,1,\cdots \tag{17}$$

then we have

Theorem 12.

Under the condition (a), (b) in Theorem 10, the numerical continuation method (16), (17) is feasible.

Proof. In fact, by the Lemma 3.1 in [1], we see that

$$G_x(t,x(t)) = I - \Gamma(x(t))f'(x(t))$$

and $G_x(t,x)$ is continuous in (t,x). Moreover, from (14), we have

$$\rho(G_x(t,x(t)) \leqq \alpha < 1,\ t \in [0,1].$$

Thus, the conditions in Theorem 11 are satisfied and the feasibility of (16), (17) is proved.

6. The implicit function theorem.

The implicit function is also an important problem in nonlinear analysis. We refer to the book [14], Theorem 5.2.4 for the general result of the theorem. In this section, the interval method will also be used to discuss this problem under weaker restrictions.

Suppose that $f(x,y): = = D_1 \times D_2 \subset R^n \times R^n \to R^n$ is continuous on an open set D and there is an identity

$$f(x,y_1) = f(x,y) + h(x,y_1,y)(y_1 - y), \quad x \in D_1,\ y_1,y \in D_2,$$

where $h(x,y_1,y) = (h(x,y_1,y)_{ij})$.
Suppose that $H(x,Y,y)$ is an inclusion monotonic interval extension of $h(x,y_1,y)$ on Y, $Y \subset Y^{(0)} \subset D_2$ with the Lipschitz condition

$$d(H(x,Y,y),\ h(x,y_1,y)) \leqq \lambda d_x(Y,y).$$

For brevity, write $H(x,Y,m(Y)) = H(x,Y)$. Consider an interval $X^{(0)} \times Y^{(0)} \subset D$ and a mapping

$$p_x(y) = y - \Gamma f(x,y),$$

where Γ is a nonsingular real matrix, then for all $x \in X^{(0)}$, $p_x(y)$ is continuous on $Y^{(0)}$ and

$$p_x(y) \in P_x(Y), \quad y \in Y,$$

where

$$P_x(Y) = m(Y) - \Gamma f(x,m(Y)) + (I - \Gamma H(x,Y))(Y-m(Y)), \tag{18}$$

Theorem 13 [20].
If for fixed $x^0 \in X \subset X^{(0)}$, there exists an interval $Y \subset Y^{(0)}$ such that

$$P_{x^0}(Y) \leqq \mathrm{Int}(Y),$$

then

(a) $f(x^0,y) = 0$ has a unique solution $y^0 = y(x^0) \in Y$;

(b) for each $y \in Y$, $H(x^0,y)$ is nonsingular;

(c) there exists an interval $X = [x^0 - \delta_1, x_0 + \delta_1]$ and an interval $Y = [y^0 - \delta_2, y^0 + \delta_2] \subset Y^0$ such that, for any $x \in X$, $f(x,y) = 0$ has a unique solution $y = y(x) \in Y$ and the mapping $y: X \to Y$ is continuous.

Proof. We can apply the preceeding results in the section 3 to prove (a) and (b). (c) follows immediately from (a), (b) and Theorem 5.2.4 in [14].

Continuating the result, we will also have a computationally verifiable sufficient condition for the global implicit function theorem. The theorem has the special feature that it does not require an initial solution point of the implicit function.

Theorem 14.

If for each $x \in D_1$ and every $y \in D_2$, there exists an interval $Y_x = [y - \frac{\delta_x(y)}{2}, y + \frac{\delta_x(y)}{2}]$ and a nonsingular real matrix $\Gamma_x(y)$, continuous in y, such that

(a) $\|\Gamma_x(y)\| \leq M$, M = constant;

(b) there exists a constant $\alpha(x)$, $0 \leq \alpha(x) < 1$ such that

$$|P_x(Y_x) - m(Y_x)| \leq \alpha(x) \frac{\delta_x(y)}{2},$$

then for each $x \in D_1$, there exists a unique solution $y(x)$ to $f(x,y) = 0$ and $y(x)$ is continuous on D_2.

The proof of this theorem is similar to that of Theorem 7, we do not give it here. □

Similarly, the following two special cases are interested.

(a) Suppose that $f(x,y)$ is local Lipschitz with respect to y. For each $x \in X^{(0)}$ and interval $Y \subset Y^{(0)}$, let

$$\partial_2 F(x,Y) = [\underline{A_{ij}}(x), \overline{A_{ij}}(x)],$$

where

$$\underline{A_{ij}}(x) = \min \{ A_{ij}(x) \mid A \in \overline{co} \bigcup_{y \in Y} \partial_2 f(x,y) \},$$

$$\overline{A_{ij}}(x) = \max \{ A_{ij}(x) \mid A \in \overline{co} \bigcup_{y \in Y} \partial_2 f(x,y) \}.$$

and $\partial_2 f(x,y)$ is the generalized derivative with respect to y. Let $P_x(Y)$ be in (18) with $H(x,Y) = \partial_2 f(x,Y)$.

(b) Suppose that $f(x,y)$ and its derivative $f_y(x,y)$ are continuous on D. Define

$$f_y(x,Y) = [\min_{y \in Y} f(x,y), \max_{y \in Y} f(x,y)]$$

and let $P_x(Y)$ be given as in (18) with $H(x,Y) = f_y(x,Y)$.

Corollary 15.

In both cases (a) and (b), Theorem 14 is valid.

Specially, if $f_x(x,y)$ exists at (x^O,y^O) and case (b) applies, then $y(x)$ is differentiable at x^O and

$$y'(x^O) = -[f_y(x^O,y^O)]^{-1} f_x(x^O,y^O).$$

Furthermore, by the differentiability assumption of $f(x,y)$, the following converse conclusion can be proved.

Theorem 16.

Suppose that $f(x,y): D \subset R^n \times R^n \to R^n$ is continuous on an open neighborhood of (x^O,y^O) for which $f(x^O,y^O) = 0$. Assume that $f_y(x,y)$ exists in a neighborhood of (x^O,y^O) and is continuous at (x^O,y^O) and that $f_y(x^O,y^O)$ is nonsingular, then there exists an interval $Y = [y^O - \frac{\delta}{2}, y^O + \frac{\delta}{2}]$ such that

$$P_{x^O}(Y) \subseteq \text{Int}(Y). \qquad (19)$$

Proof. Let $\Gamma = [f_y(x^O,y^O)]^{-1}$, then

$$P_{x^O}(Y) = y(x^O) - \Gamma f(x^O,y(x^O)) + (I - \Gamma f(x^O,Y))(Y - y(x^O)).$$

For any $y \in P_{x^O}(Y)$, we have

$$y = \beta + \mu$$

with $\beta = y(x^O) - \Gamma f(x^O,y(x^O)) = y(x^O)$ and $\mu \in (I - \Gamma fy(x^O,Y))(Y - y(x^O))$.
Since

$$\begin{aligned}\|I - \Gamma f_y(x^O,Y)\| &\leqq \|\Gamma(f_y(x^O,y^O) - f_y(x^O,Y))\| \\ &\leqq \|\Gamma\|\|f_y(x^O,y^O) - [\inf_{y\in Y} f_y(x^O,y), \sup_{y\in Y} f_y(x^O,y)]\| \\ &\leqq \|\Gamma\|\max\{\inf_{y\in Y}(f_y(x^O,y^O) - f_y(x^O y)), \\ &\qquad \sup_{y\in Y}(f_y(x^O,y^O) - f_y(x^O,y))\}\end{aligned}$$

and by the continuity of $f_y(x,y)$ on D, there exists δ, such that for $y \in Y = [y^O - \frac{\delta}{2}, y^O + \frac{\delta}{2}]$, we have

$$\|I - \Gamma f_y(x^O,Y)\| < 1$$

and

$$\|(I - \Gamma f_y(x^O,Y))(Y - y^O)\| < \frac{\delta}{2},$$

therefore

$$\|P_{x^O}(Y) - y^O\| < \frac{\delta}{2}$$

and (18) is proved.

□

Under the differentiability assumption of $f(x,y)$, we can also obtain a stronger global implicit function theorem.

Theorem 17.

If for each fixed $x \in D$, and every $y \in D_2$, there exists an interval $Y_x = [y - \frac{\delta_x(y)}{2}, y + \frac{\delta_x(y)}{2}]$ and a nonsingular real matrix $\Gamma_x(y)$, continuous in y such that

(a) $\|\Gamma_x(y)\| < \omega_x(\|y\|)$, $y \in D_2$, $\omega_x(t) \in \Omega$;

(b) there exists a constant $\alpha(x)$, $0 \leq \alpha(x) < 1$ such that

$$|P_x(Y_x) - m(Y_x)| < \alpha(x) \frac{\delta_x(y)}{2} ,$$

then

(a) for each $x \in D_1$, there exists a unique solution $y(x)$ to $f(x,y) = 0$ and $y(x)$ is continuous on D_1;

(b) if $f_x(x,y)$ exists and is continuous on D, then

$$y'(x) = -[f_y(x,y)]^{-1} f_x(x,y).$$

In connection with the theorem of M. and S. Radulescu [17], we have the following sufficient condition for Theorem 17.

<u>Theorem 18</u>.

Let $\omega \in \Omega$, $f(x,y)\colon D = D_1 \times D_2 \subset R^n \times R^n \to R^n$ be continuously differentiable on D and for all $(x,y) \in D$, and

(a) $f_y(x,y)$ be an isomorphism from D_2 on D_2;

(b) $\|f_y(x,y)^{-1}\| (1 + \|f_x(x,y)\|) \leq \omega(\max(\|x\|, \|y\|)$,

then the condition of Theorem 17 will be hold with

$$\omega_x(\|y\|) = \begin{cases} \omega(\|x\|), & \text{for } \|x\| \geq \|y\|; \\ \omega(\|y\|), & \text{for } \|y\| \geq \|x\|. \end{cases}$$

Since the proofs of Theorem 17 and 18 are similar to those of Theorem 10 and 16, respectively, we do not give them, too.

7. Acknowledgments

I wish to express my deep gratitute to Prof. Dr. K. Nickel for his financial support which made it possible for me to participate the International Interval Symposium in Freiburg, 1985.

I am also thankful to Professors K. Nickel, R.E. Moore, Dr. A. Neumaier and Dr. J. Garloff for their invaluable help and advices, and also to Mrs. L. Wüst for her excellent typewritting.

References

[1] Avila, J.H.: The feasibility of continuation methods for nonlinear equations, SIAM. J. Numer. Anal. 11 (1974).

[2] Krawczyk, R.: Newton-Algorithmen zur Bestimmung von Nullstellen mit Fehlerschranken, Computing 4, 187-201 (1969).

[3] Krawczyk, R.: Zentrische Formen und Intervalloperatoren, Freiburger Intervall-Berichte 82/1, 1-30 (1982).

[4] Krawczyk, R.: Intervallsteigungen für rationale Funktionen und zugeordnete zentrische Formen. Freiburger Intervall-Berichte 83/2, 1-30 (1983).

[5] Hansen, E.R.: The centred form. in: Topics in Interval Analysis, (E.R. Hansen, Ed.) Oxford, pp. 102-105.

[6] Mihai Cristea: A note on global inversion theorems and applications to differential equations. Nonlinear Analysis, Theory, Methods and Applic., pp. 1155-1161 (1981).

[7] Moore, R.E.: Interval Analysis. Prentice-Hall, Englewood Cliffs, NJ (1966).

[8] Moore, R.E.: A test for existence of solution to nonlinear systems, SIAM J. Numer. Anal. 14, 611-615 (1977).

[9] Moore, R.E.: Methods and Applications of Interval Analysis. SIAM Publications, Philadelphia (1974).

[10] Moore, R.E., and Qi, L.: A successive interval test for nonlinear systems, SIAM J. Numer. Anal. 19 (1982).

[11] Moore, R.E., and Zuhe Shen: An interval version of Chebyshev's method for nonlinear operator equations. Nonlinear Analysis, Theory, Methods and Applic. 7, 21-34 (1983).

[12] Neumaier, A.: Interval norms, Freiburger Intervall-Berichte 81/5, 17-33 (1981).

[13] Nickel, K.: On the Newton method in interval analysis. MRC Report No. 1136, Mathematics Research Center. University of Wisconsin-Madison, (1971).

[14] Ortega, J.M., and Rheinboldt, W.C.: Iterative Solution of Non-linear Equations in Several Variables, Academic Press, New York-London (1970).

[15] Plastock. R.: Homeomorphism between Banach spaces. Trans. Amer. Math. Soc., 200, 169-183 (1974).

[16] Pourciau, B.H.: Analysis and optimization of Lipschitz continuous mappings. J. Optim. Theory Appl. 22, 311-351 (1977).

[17] Radulescu, M., and Radulescu, S.: Global inversion theorem and applications to differential equations. Nonlinear Analysis, Theory, Methods and Applic. 4, 951-965 (1980).

[18] Ratschek, H.: Zentrische Formen. ZAMM 58, 434-436 (1978).

[19] Schwetlick, H.: Numerische Lösung nichtlinearer Gleichungen, VEB Deutscher Verlag des Wissenschaften, Berlin, 1978.

[20] Zuhe, Shen: On some classical existence theorems. Nonlinear Analysis, Theory, Methods and Applic. 7, 1024-1033 (1983).

[21] Zuhe, Shen: Diffeomorphism and the feasibility of the numerical continuation methods. Nonlinear Analysis, Theory, Methods and Applics. 9, 495-502 (1985).