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Solving the linear interval tolerance problem

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Abstract

For the interval linear system $\mathbf{A}x = \mathbf{b}$, the *linear tolerance problem* is considered that requires inner evaluation of the *tolerable solution set* $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(Ax \in \mathbf{b})\}$ formed by all point vectors x such that the product Ax remains within $\mathbf{b}$ for all possible $A \in \mathbf{A}$. Along with the simple incompatibility criterion, we develop comprehensive solvability theory for the linear tolerance problem that not only settles whether $\Sigma_{\forall\exists}$ is empty or not, but also enables modification of the problem to ensure its desired properties. To conclude, we advance several numerical methods of various accuracy and complexity for construction of an interval solution to the linear tolerance problem around a given center.

1. Introduction

This paper deals with both mathematical and computational aspects of the linear static systems under uncertainty. However, we shall not consider them in the context of probabilistic or fuzzy models as is fashionable among modern system analysts. The uncertainty our paper is devoted to is *interval*, that is, the bounds of possible variations of the parameters, both those prescribed by our will and those resulted from our ignorance, are the only information about the system we have at our disposal. If need be, one may take as a fact that the parameters of the system have *bounded* uncertainty (similar to what was done in the works [8,33,34]), but it is described by interval analysis tools in our case.

The natural mathematical model for deterministic linear static systems is the linear algebraic equation

$$Ax = b, \tag{1}$$

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where A is an $m \times n$-matrix and b is an m-vector. We shall restrict our attention to the real case and assume that we only know that the elements of (1) may independently vary within the intervals $[\underline{a}_{ij}; \bar{a}_{ij}] = \mathbf{a}_{ij}$ and $[\underline{b}_i; \bar{b}_i] = \mathbf{b}_i$ respectively. Put otherwise, we formally have the interval system of linear algebraic equations

$$\mathbf{A}x = \mathbf{b} \tag{2}$$

with an interval $m \times n$-matrix $\mathbf{A}$ and an interval right-hand side m-vector $\mathbf{b}$. The solution set to (2) has been defined in a variety of ways: aside from the *united solution set*

$$\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\}$$

commonly used in applications there exists, for example, the *controllable solution set*

$$\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall b \in \mathbf{b})(\exists A \in \mathbf{A})(Ax = b)\}$$

(see [26]) among many others. But the subject matter of our paper will be the *tolerable solution set* formed by all point vectors x such that the product Ax falls into $\mathbf{b}$ for any $A \in \mathbf{A}$, i.e., the set

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \Sigma_{\forall\exists} = \{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\}, \tag{3}$$

sometimes written out as

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(Ax \in \mathbf{b})\}.$$

Neumaier in [10,11] introduced the term *restricted solution set* for (3), and other authors followed; they denote the set $\Sigma_0(\mathbf{A}, \mathbf{b})$. Others speak of "inner solutions", but I prefer the more adequate term — *tolerable* — that used in Russian works. The history of the set (3) and of some related problems was described comprehensively in the papers by Neumaier [10] and by Kelling and Oelschlägel [4].

To clarify what the tolerable solution set has to do with the mathematical modelling and the system analysis, it is very instructive to consider its practical interpretation. Let the "black box" be given with the input subjection vector $x \in \mathbb{R}^n$ and the output response vector $y \in \mathbb{R}^m$, where the input-output relationship is linear, i.e., $y = Ax$ with a real $m \times n$-matrix $A = (a_{ij})$. Suppose that the parameters of the black box are not precisely known, but are given only by intervals $\mathbf{a}_{ij}$, $a_{ij} \in \mathbf{a}_{ij}$, which constitute the interval $m \times n$-matrix $\mathbf{A} = (\mathbf{a}_{ij})$. For example, these parameters may vary in an unpredictable way (drift) within $\mathbf{a}_{ij}$, or the interval uncertainty may be intrinsic to the very description of the mathematical model.

Also assume that the set of the black box output states is specified as an interval vector $\mathbf{y}$ and we must ensure y to arrive at it no matter what the exact values of a_{ij} from $\mathbf{a}_{ij}$ are (see Fig. 1). Our interest is in finding input signal $\tilde{x}$ such that for any values of the parameters a_{ij} from $\mathbf{a}_{ij}$

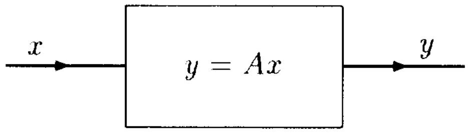

Fig. 1. A model for interpretation of the tolerable solution set.

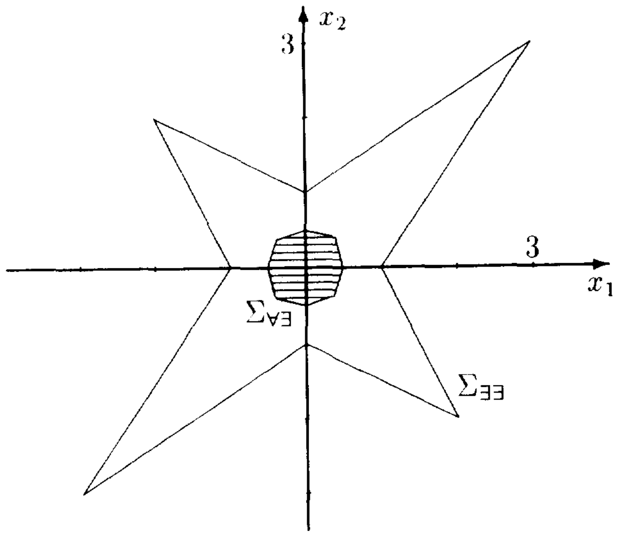

Fig. 2. The solution sets to (5).

we altogether get the output response y within the required tolerances $\mathbf{y}$. The tolerable solution set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{y})$ is just the set of all such $\tilde{x}$'s. That general scheme is known to be successfully applied to specific problems in mathematical economics by Rohn [20,21], in automatic control by Khlebalin [5,6], Skybytsky and Yuping [32], Zakharov and Shokin [35] and so on.

The ideas developed above are straightforwardly transferred to the general case of interval nonlinear system

$$F(x, \mathbf{a}) = \mathbf{b}, \tag{4}$$

with $F(x, a) = (f_1(x, a), f_2(x, a), \ldots, f_m(x, a))$ and $\mathbf{a}$, $\mathbf{b}$ being interval vectors of the same dimension as a, b, respectively. Let us call

$$\Sigma_{\forall\exists}(F; \mathbf{a}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall a \in \mathbf{a})(\exists b \in \mathbf{b})(F(x, a) = b)\}$$

the *tolerable solution set* to the interval system (4). If for the considered "black box" the input-output relationship has the form $y = F(x, a)$, then the regimes that correspond to the points of $\Sigma_{\forall\exists}(F; \mathbf{a}, \mathbf{y})$ (and no one else) ensure functioning of the device within the required output state corridor $\mathbf{y}$ irrespective of the input subjection a from $\mathbf{a}$. In our work, however, we shall study only the linear case at length.

In general, the tolerable solution set to interval linear system is easily proved (for instance, in [4,22,23] or in Section 3 of our paper) to be a convex polyhedral set in $\mathbb{R}^n$. The Fig. 2 depicts, for example, the set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ for the system

$$\begin{pmatrix} [1; 2] & \left[-\frac{2}{3}; \frac{1}{2}\right] \\ \left[-\frac{2}{3}; \frac{1}{2}\right] & [1; 2] \end{pmatrix} x = \begin{pmatrix} [-1; 1] \\ [-1; 1] \end{pmatrix}, \tag{5}$$

from the work [27]. Nevertheless, if the dimension of the interval system is large, then the direct description of its tolerable solution set becomes laborious and practically useless (its complexity is proportional to $m \cdot 2^n$). For this reason it is expedient to confine ourselves to finding some simple subsets $\chi \subseteq \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$, since for any $x \in \chi$ the characteristic condition

$$(\forall A \in \mathbf{A})(Ax \in \mathbf{b})$$

remains valid. In other words, we change $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ by its inner approximation, usually formulating the problem to be solved in the following form:

Find an interval vector that is contained in the tolerable solution set (if nonempty) of the interval linear system. (6)

This *linear tolerance problem* (LTP) is the subject of the present paper, and its principal practical significance lies in the fact that it is a generalized *stabilization problem* for the static systems with disturbances.

Sometimes (6) is referred to as *inner problem* for interval linear systems [2,23,25] and the points of the tolerable solution set as *inner solutions* [22,32]. The main mathematical results of our work are new techniques for the investigation of solvability of the linear tolerance problem (Sections 3–5) as well as methods for inner approximation of the tolerable solution set (Sections 6–8). Previously, some results of the present paper have been published in the separated and abridged form in [23–25,28].

2. Discussion of the problem statement

In this paper, intervals and other interval objects are denoted by boldface letters, for example, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{x}, \mathbf{y}, \mathbf{z}$, while non-interval (real) objects are not distinguished in any way. Also, we need the following notation:

$\mathbb{IR}$ — the set of all real intervals $[\underline{x}; \overline{x}]$ on $\mathbb{R}$, $\underline{x} \leqslant \overline{x}$,
$\mathbb{IR}^n$ — the set of n-dimensional interval vectors,
int X — topological interior of the set X in $\mathbb{R}^n$ with the standard topology,
$\overline{\mathbf{x}}$, $\underline{\mathbf{x}}$ — upper and lower bounds of the interval $\mathbf{x}$, respectively,
$\operatorname{mid} \mathbf{x} = (\overline{\mathbf{x}} + \underline{\mathbf{x}})/2$ — mean value (midpoint) of the interval $\mathbf{x}$,
$\operatorname{rad} \mathbf{x} = (\overline{\mathbf{x}} - \underline{\mathbf{x}})/2$ — radius of the interval $\mathbf{x}$,
vert $\mathbf{x}$ — vertex set of the interval (interval vector, matrix) $\mathbf{x}$,
$|\mathbf{x}| = \max\{|\overline{\mathbf{x}}|, |\underline{\mathbf{x}}|\}$ — absolute value (magnitude) of the interval $\mathbf{x}$,
$\langle \mathbf{x} \rangle = \begin{cases} \min\{|\overline{\mathbf{x}}|, |\underline{\mathbf{x}}|\}, & \text{if } 0 \notin \mathbf{x}, \\ 0, & \text{otherwise} \end{cases}$ — mignitude of the interval $\mathbf{x}$ or the least distance between points of $\mathbf{x}$ and zero, in some sense the opposite of the absolute value.

If $\mathbf{x} = (\mathbf{x}_i)_{i=1}^n$ is an interval vector, then all of the operations defined above are to be understood componentwise, so that rad $\mathbf{x}$, for instance, is the real vector $(\operatorname{rad} \mathbf{x}_i)_{i=1}^n$.

The interval $\mathbf{x}$ is said to be *symmetric* if mid $\mathbf{x} = 0$, that is, if $\overline{\mathbf{x}} = -\underline{\mathbf{x}}$.

Throughout the rest of this paper, all arithmetic operations with intervals and interval objects are those of classical interval arithmetic [1–3,9,11,17]:

$$[\underline{x}; \overline{x}] + [\underline{y}; \overline{y}] = [\underline{x} + \underline{y}; \overline{x} + \overline{y}],$$

$$[\underline{x}; \overline{x}] - [\underline{y}; \overline{y}] = [\underline{x} - \overline{y}; \overline{x} - \underline{y}],$$

$$[\underline{y};\bar{x}]\cdot[\underline{y};\bar{y}]=\left[\min\{\underline{x}\,\underline{y},\underline{x}\,\bar{y},\bar{x}\,\underline{y},\bar{x}\,\bar{y}\};\max\{\underline{x}\,\underline{y},\underline{x}\,\bar{y},\bar{x}\,\underline{y},\bar{x}\,\bar{y}\}\right],$$

$$[\underline{x};\bar{x}]/[\underline{y};\bar{y}]=[\underline{x};\bar{x}]\cdot[1/\bar{y};1/\underline{y}]\quad\text{for}[\underline{y};\bar{y}]\not\ni 0.$$

Besides, the endwise interval subtraction

$$[\underline{\mathbf{x}};\bar{\mathbf{x}}]\ominus[\underline{\mathbf{y}};\bar{\mathbf{y}}]=[\underline{\mathbf{x}}-\underline{\mathbf{y}};\bar{\mathbf{x}}-\bar{\mathbf{y}}]$$

will be necessary in our work. The result of this operation is an interval only if rad $\mathbf{x}\geqslant$ rad $\mathbf{y}$.

Notice that the tolerable solution set may turn out to be empty even for "good" interval data, as, for instance, it does in the one-dimensional case $\mathbf{A}=[2;3]$, $\mathbf{b}=[1;2]$. The two-dimensional system

$$\begin{pmatrix}[1;2] & [-1;1]\\ [-1;1] & [1;2]\end{pmatrix}x=\begin{pmatrix}[1;3]\\ [1;3]\end{pmatrix}\tag{7}$$

gives a more complex example with the empty tolerable solution set. In such cases, we shall speak that the linear tolerance problem is *unsolvable* (incompatible), since then the initial problem statement (6) becomes empty. In our paper, much attention will be accordingly paid to development of the technique for detailed solvability examination and correction (Sections 3–5).

It is pertinent to note that the inclusion of a vector x to the tolerable solution set is equivalent to

$$\mathbf{A}\cdot x\subseteq\mathbf{b},$$

where "·" means the standard interval matrix-vector multiplication, since $\{Ax\mid A\in\mathbf{A}\}=\mathbf{A}\cdot x$ [1,11]. The equality

$$\{Ax\mid A\in\mathbf{A},\ x\in\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})\}=\mathbf{b}$$

may appear to be valid. This is not so, however, as the tolerance problem with the data $\mathbf{A}=[-1;1]$, $\mathbf{b}=[0;1]$ shows. Here, only zero may belong to $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})$, since otherwise, multiplying $y\neq 0$, $y\in\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})$ on $(-\operatorname{sgn} y)$ from the interval $[-1;1]=\mathbf{A}$, we would get a negative number $-|y|\notin[0;1]$. Therefore,

$$\{Ax\mid A\in\mathbf{A},\ x\in\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})\}=0\neq[0;1]=\mathbf{b}.$$

On its turn, if a set $\mathscr{E}\subseteq\mathbb{R}^n$ has the property

$$\{Ax\mid A\in\mathbf{A},\ x\in\mathscr{E}\}=\mathbf{b},\tag{8}$$

this does not necessarily imply that $\mathscr{E}=\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})$. To illustrate, let us consider the one-dimensional example of the linear tolerance problem with $\mathbf{A}=\mathbf{b}=[-1;1]$. Now $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})=[-1;1]$, but for each of intervals $[\kappa;1]$ or $[-1;\kappa]$, $-1\leqslant\kappa\leqslant 1$, (8) holds too. Hence, the property (8) does not entirely characterize the tolerable solution set of interval linear systems, but it is fairly simple to realize that $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})$ is the most inclusive of the sets $\mathscr{E}$ satisfying

$$\{Ax\mid A\in\mathbf{A},\ x\in\mathscr{E}\}\subseteq\mathbf{b}.$$

One of the classical interval analysis problem that has numerous and significant applications is known to be the *outer problem*, that is, the problem of outer component wise estimation of the united solution set

$$\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\},$$

the set of solutions to all point linear systems $Ax = b$ contained in (2) (see e.g. [1,2,9,11,27] and the extensive references there). Usually, it is formulated as follows:

Find an interval vector that contains the united solution set of the interval linear system.

The inclusion

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\}$$
$$\subseteq \{x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\} = \Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})$$

is quite obvious and so, if the interval vectors $\mathbf{U}$ and $\mathbf{V}$ are solutions, respectively, to the linear tolerance problem and to the outer problem for the interval linear system, then

$$\mathbf{U} \subseteq \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \subseteq \Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b}) \subseteq \mathbf{V},$$

that is, each solution of the linear tolerance problem is contained in every solution of the outer problem. Fig. 2 demonstrates that the sets $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ and $\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})$ may greatly differ in size and this is especially telling for the corresponding interval solutions $\mathbf{U}$ and $\mathbf{V}$.

In general, the tolerable solution set may be unbounded, but if the interval system is square, that is, $m = n$, and at least one point matrix $\tilde{A} \in \mathbf{A}$ is nonsingular, then $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ is a bounded set, since

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \subseteq \{\tilde{A}^{-1}b \mid b \in \mathbf{b}\}.$$

In this case the solution process for the linear tolerance problem has a pictorial geometrical interpretation, as inscribing a rectangular box with the sides parallel to the coordinate axes in the convex polyhedron $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$.

Although throughout this paper we consider the inner approximation of the tolerable solution set by interval vectors, that is, by rectangular boxes whose sides are parallel to the coordinate axes, one should be fully aware of possible advantages and disadvantages of such a way of estimation. It may turn out bad (see Fig. 3) in the sense that the ratio of volumes of $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ and of its best inner interval approximations can be arbitrary large (for ill-conditioned $\mathbf{A}$).

It is amply clear that

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \bigcap_{A \in \mathbf{A}} \Sigma_{\forall\exists}(A, \mathbf{b}), \tag{9}$$

and

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \bigcap_{A \in \mathbf{A}} \{A^{-1}b \mid b \in \mathbf{b}\}, \tag{10}$$

if $\mathbf{A}$ is square and nonsingular, that is, all point matrices $A \in \mathbf{A}$ are nonsingular. Though the definition of the tolerable solution set requires the product Ax to get into the right-hand side

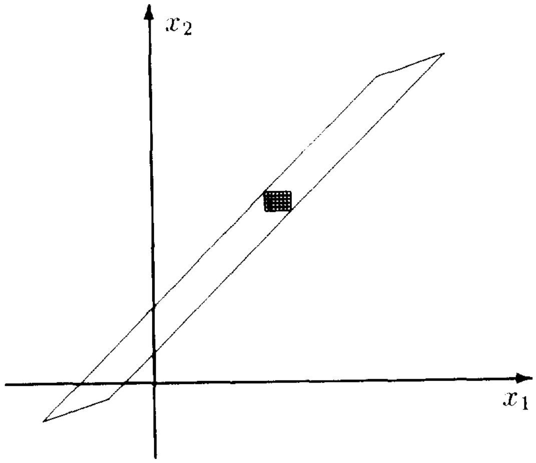

Fig. 3. The inner estimation by interval vectors may turn out bad.

vector **b** for *every* $A \in \mathbf{A}$, we show that in the linear case it is sufficient for the inclusion $Ax \in \mathbf{b}$ to be fulfilled for each A from a *finite* subset of matrices within **A**. Namely, there holds

Lemma 1. $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall A \in \text{vert } \mathbf{A})(Ax \in \mathbf{b})\}$.

It too implies in particular that

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \bigcap_{A \in \text{vert } \mathbf{A}} \Sigma_{\forall\exists}(A, \mathbf{b}), \tag{11}$$

and if **A** is square and all point matrices $A \in \text{vert } \mathbf{A}$ are nonsingular (and not necessarily all $A \in \mathbf{A}$),

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \bigcap_{A \in \text{vert } \mathbf{A}} \{A^{-1}b \mid b \in \mathbf{b}\},$$

that is, we substantially refine the representations (9) and (10).

Proof of the Lemma actually boils down to the verification of whether

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \supseteq \{x \in \mathbb{R}^n \mid (\forall A \in \text{vert } \mathbf{A})(Ax \in \mathbf{b})\},$$

since the inverse inclusion is obvious.

Suppose some vector $x \in \mathbb{R}^n$ satisfies

$$Ax \in \mathbf{b}$$

for all $A \in$ vert **A**. Let E be a matrix from **A**. According to the definition of vert **A**, there exist coefficients $\lambda_A \geqslant 0$, with their number equal to 2^{mn}, such that

$$\sum_{A \in \text{vert } \mathbf{A}} \lambda_A = 1 \quad \text{and} \quad E = \sum_{A \in \text{vert } \mathbf{A}} \lambda_A A,$$

or, put otherwise, E is represented as a convex combination of the extreme matrices from $\mathbf{A}$. Then

$$Ex = \Big(\sum_{A \in \text{vert}\, \mathbf{A}} \lambda_A A \Big) \cdot x = \sum_{A \in \text{vert}\, \mathbf{A}} \lambda_A . Ax \qquad (12)$$

But all $Ax \in \mathbf{b}$ in virtue of the statement of the Lemma. Therefore, their convex combination, such as the sum (12) is, also belongs to the convex set $\mathbf{b}$. □

In this work, we shall follow the approach to the linear tolerance problem that may be called "center": first one finds a point of the interior int $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ — the "center" — and then, if succeeded, an interval solution is constructed around it. There exists the other, "algebraic", approach to the linear tolerance problem, first formulated by Zyuzin [36] (though in the very vague form). Its essence is the change of the original linear tolerance problem for the problem of finding an algebraic interval solution to the interval system, that is, the interval vector $\mathbf{x}_a$ such that its substitution into the system (2) and execution of all interval arithmetic operations results to the valid equality $\mathbf{A}\mathbf{x}_a = \mathbf{b}$. Then, due to inclusion monotonicity of the interval vector-matrix operations [1,11], there holds

$$\mathbf{A}x \subseteq \mathbf{A}\mathbf{x}_a = \mathbf{b}$$

for any $x \in \mathbf{x}_a$, and so $\mathbf{x}_a \subseteq \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$. The linear tolerance problem thus reduces to a purely algebraic one: solve a system of equations in the interval space $\mathbb{IR}^n$. This is a very attractive circumstance, notwithstanding one can not perform complete solvability examination for the linear tolerance problem using the algebraic approach (see the next Section). The numerical procedures initially proposed to implement the algebraic approach in [35–37] were not sufficiently elaborated, but recently Shary [29,30] has advanced very efficient computational algorithm based on imbedding of the equation into Kaucher extended interval arithmetic and further application of *subdifferential Newton method*. It is intended primarily for quick calculation of the solutions to (6) when its "good" solvability is given *a priori* (in real-time devices, for example).

3. Quick examination of solvability

The results concerning solvability of the linear tolerance problem have been appearing long ago in the publications on the subject. Rohn turned to the linear tolerance problem in [20,21] when studying linear economic input-output models (the interval Leontiev-type equation). In those his works, explicit formulae are written out that enable to examine solvability of the linear tolerance problem, but for a special type of the interval matrix $\mathbf{A}$ and nonnegative $\mathbf{b}$. In the work by Khlebalin [5] as well as in [10,23], the following simple heuristic LTP solvability test was proposed: the solution $\tilde{x}$ of the "middle" point system

$$\text{mid}\, \mathbf{A} \cdot x = \text{mid}\, \mathbf{b}$$

is taken as the most probable representative of the tolerable solution set, and if $\mathbf{A}\tilde{x} \not\subseteq \mathbf{b}$ we infer "practical unsolvability" of the linear tolerance problem, though, strictly speaking, no definite conclusion may be done in this case. This criterion is easily seen to work only when the matrix **A** is "sufficiently narrow" as compared with the right-hand side vector **b** and is not able to distinguish boundary situations. The examples below demonstrating its fault were first presented by the author in [25].

Let $\mathbf{A} = [-1; 2]$, $\mathbf{b} = [-2; 6]$. Then $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = [-1; 2]$, but the solution to the middle system is 3 and it does not belong to the tolerable solution set. The two-dimensional counterexample is very interesting, with the data

$$\mathbf{A} = \begin{pmatrix} 3 & [1; 2] \\ [1; 2] & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} [5; 7] \\ [7; 9] \end{pmatrix}.$$

Here $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ consists of the single point $(1, 2)^{\mathrm{T}}$ while the solution of the "middle system" is $(\frac{8}{9}, \frac{20}{9})^{\mathrm{T}}$. The feature of this example is that the matrix of the problem is strictly positive and contains only nonsingular point matrices.

In algebraic approach to the linear tolerance problem developed in [30,35,36], the conclusion on whether $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ is empty or not is taken while constructing an algebraic interval solution to the interval system. Sometimes this enables to recognize solvability of problems for which the "middle system" test fails. Unfortunately, the algebraic interval solution does not need to exist even when the linear tolerance problem for the original interval system is compatible. It is illustrated by that same one-dimensional example with $\mathbf{A} = [-1; 2]$, $\mathbf{b} = [-2; 6]$. The algebraic interval solution of the equation $[-1; 2] \cdot x = [-2; 6]$ does not exist, but $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = [-1; 2] \neq \emptyset$.

As we have already mentioned, the tolerable solution set is a convex polyhedral set in $\mathbb{R}^n$, the equations of bounding hyperplanes being straightforwardly written out explicitly. Hence, the set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ may be represented as the set of all feasible solutions of a linear programming problem, and the question of whether it is empty or not (that is, solvability of LTP) can be resolved through applying the initial stage of the standard simplex-algorithm (entering into the basis). The corresponding linear program was first presented by Rohn [22]. Afterward, Khlebalin [6] came to the similar results, having reduced inscribing the maximal perimeter hyperbar included in $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ to the solution of a full linear programming problem.

We shall conduct comprehensive investigation of solvability of the linear tolerance problem and of some other related questions in the next sections, using a special "identifying functional" technique. At the same time, that approach, though the most informative, requires much computational labor. The aim of this section is to give a simple sufficient unsolvability criterion for the linear tolerance problem, based on comparison of the relative narrowness of elements of its interval matrix and right-hand side vector. It is intended for the preliminary quick examination of the linear tolerance problem under solution.

First, note that if the i-th row of **A** contains only zero elements, it is necessary that $\mathbf{b}_i \ni 0$ for the tolerable solution set to be nonempty. If this condition holds, then the property of $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ being empty or nonempty depends upon the other, not the i-th, rows of **A** and components of **b**. Thus, without loss of generality, we may assume in the rest of this paper that **A** does not have zero rows.

To characterize "relative narrowness" of nonzero intervals, Ratschek introduced in [16] the functional

$$\chi(\mathbf{x}) = \begin{cases} \underline{\mathbf{x}}/\overline{\mathbf{x}}, & \text{if } |\underline{\mathbf{x}}| \leqslant |\overline{\mathbf{x}}|, \\ \overline{\mathbf{x}}/\underline{\mathbf{x}}, & \text{otherwise}. \end{cases}$$

Clearly, $-1 \leqslant \chi(\mathbf{x}) \leqslant 1$, and $\chi(\mathbf{x}) = 1$ if and only if $\mathbf{x} \in \mathbb{R}$. Moreover, it is proved in [16] that

$$\chi(\mathbf{x}) = \chi(\mathbf{y}) \text{ if and only if } \mathbf{x} = \lambda \mathbf{y},\ \lambda \in \mathbb{R},\quad \lambda \neq 0, \tag{13}$$

$$\text{if } \mathbf{x} + \mathbf{y} \neq 0, \text{ then } \chi(\mathbf{x} + \mathbf{y}) \leqslant \max\{\chi(\mathbf{x}), \chi(\mathbf{y})\}, \tag{14}$$

$$\text{if } \mathbf{x} \supseteq \mathbf{y} \text{ and } \chi(\mathbf{y}) \geqslant 0, \text{ then } \chi(\mathbf{x}) \leqslant \chi(\mathbf{y}). \tag{15}$$

Now we are able to formulate and to prove the

Theorem 1. *Let the interval $m \times n$-matrix* $\mathbf{A}$ *and the interval m-vector* $\mathbf{b}$ *be such that for some $k \in \{1, 2, \dots, m\}$ the following conditions hold*:
(i) $0 \notin \mathbf{b}_k$,
(ii) $\max\{\chi(\mathbf{a}_{kj}) \mid 1 \leqslant j \leqslant n,\ \mathbf{a}_{kj} \neq 0\} < \chi(\mathbf{b}_k)$.
Then the tolerable solution set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ is empty.

For example, using this criterion one can verify that the one-dimensional system with $\mathbf{A} = [2; 3]$, $\mathbf{b} = [1; 2]$ considered in the beginning of the Section 2 has empty tolerable solution set.

Proof of the Theorem [24] will be conducted *ad absurdum* employing a technique similar to that developed in [18]. Let us assume that the tolerance problem nonetheless has a solution $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \neq \emptyset$, that is, $\mathbf{A}t \subseteq \mathbf{b}$, the condition (i) making it impossible for the interval $(\mathbf{A}t)_k$ to equal zero. Then the following inequalities are true:

$$\begin{aligned} \chi((\mathbf{A}t)_k) &= \chi\left(\sum_{j=1}^{n} a_{kj} t_j\right) \\ &\leqslant \max\{\chi(\mathbf{a}_{kj} t_j) \mid 1 \leqslant j \leqslant n,\ \mathbf{a}_{kj} t_j \neq 0\} \qquad \text{by (14)} \\ &= \max\{\chi(\mathbf{a}_{kj}) \mid 1 \leqslant j \leqslant n,\ \mathbf{a}_{kj} t_j \neq 0\} \qquad \text{by (13)} \\ &\leqslant \max\{\chi(\mathbf{a}_{kj}) \mid 1 \leqslant j \leqslant n,\ \mathbf{a}_{kj} \neq 0\}. \end{aligned}$$

We have found

$$\chi((\mathbf{A}t)_k) \leqslant \max\{\chi(\mathbf{a}_{kj}) \mid 1 \leqslant j \leqslant n,\ \mathbf{a}_{kj} \neq 0\}. \tag{16}$$

On the other hand, by virtue of our assumption, $(\mathbf{A}t)_k \subseteq \mathbf{b}_k$ which because of (15) implies

$$\chi((\mathbf{A}t)_k) \geqslant \chi(\mathbf{b}_k).$$

Combining this with (16) now gives

$$\max\{\chi(\mathbf{a}_{kj}) \mid 1 \leqslant j \leqslant n,\ \mathbf{a}_{kj} \neq 0\} \geqslant \chi(\mathbf{b}_k)$$

which is contrary to (ii). □

The importance of all the conditions of Theorem 3 may be exhibited on the one-dimensional example with $\mathbf{A} = [-1; 2]$, $\mathbf{b} = [-2; 6]$ mentioned above. Here $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = [-1; 2] \neq \emptyset$, though $\chi(\mathbf{A}) = -1/2 < -1/3 = \chi(\mathbf{b})$. The more profound explanation is that the property (15) of the functional χ does not hold for intervals containing zero in their interiors:

$$[-1; 1] \subseteq [-1; 2] \subseteq [-2; 2], \quad \text{but } \chi([-1; 1]) = \chi([-2; 2]) = -1, \ \chi([-1; 2]) = -\frac{1}{2}.$$

At the same time, if the conditions of Theorem 3 fail, this does not necessarily mean compatibility of the linear tolerance problem. For instance, (ii) is not true for the system (7), but even so its tolerable solution set is empty.

If one concludes from Theorem 3 that some LTP is incompatible, then

$$\Omega = \min_{1 \leqslant i \leqslant m} \left\{ \max_{1 \leqslant j \leqslant n} \chi(\mathbf{a}_{ij}) - \chi(\mathbf{b}_i) \right\} \leqslant 0,$$

and this value to some extent characterizes the degree of unsolvability of the tolerance problem: it is less, the farther problem from solvable, and vice versa. Besides, the indices $k \in \{1, 2, \ldots, m\}$ for which the condition (ii) of Theorem 3 fails point to those rows of the matrix $\mathbf{A}$ and the respective components of $\mathbf{b}$ that make the dominant contribution to the incompatibility of the given LTP. To lessen its deflection from a solvable one (to approach to solvability), one should either narrow the widest elements of these rows of the matrix $\mathbf{A}$, that is, to increase $\max_{1 \leqslant j \leqslant n} \chi(\mathbf{a}_{kj})$, or to widen the right-hand side, that is, to decrease $\chi(\mathbf{b}_k)$.

4. Detailed examination of solvability

The basis of the solvability theory developed below for the linear tolerance problem is a new analytical characterization of the tolerable solution set. Along this lines, the most important result was obtained by Rohn who had shown in [22] that $x \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ is equivalent to

$$|\text{mid}\,\mathbf{A} \cdot x - \text{mid}\,\mathbf{b}| \leqslant \text{rad}\,\mathbf{b} - \text{rad}\,\mathbf{A} \cdot |x|$$

(analogue of Oettli-Prager criterion for the united solution set [11]). In [13], Nuding gave the other proof of this statement. But the starting point of our considerations is

Lemma 2. *Let an interval $m \times n$ matrix $\mathbf{A}$ and an interval right-hand side m-vector $\mathbf{b}$ be given, so the expression*

$$\text{Tol}(x) = \text{Tol}(x; \mathbf{A}, \mathbf{b}) = \min_{1 \leqslant i \leqslant m} \left\{ \text{rad}\,\mathbf{b}_i - \left| \text{mid}\,\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \right\}$$

defines a functional $\text{Tol}\colon \mathbb{R}^n \to \mathbb{R}$. *Then the inclusion $x \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ is equivalent to* $\text{Tol}(x; \mathbf{A}, \mathbf{b}) \geqslant 0$, *i.e., the tolerable solution set of the relevant interval system is the Lebesgue set* $\{x \in \mathbb{R}^n \mid \text{Tol}(x; \mathbf{A}, \mathbf{b}) \geqslant 0\}$ *of the functional Tol.*

Proof. As we have previously stated, $x \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ holds iff $\mathbf{A}x \subseteq \mathbf{b}$. We rewrite the latter in the following form

$$\operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \subseteq [-\operatorname{rad} \mathbf{b}_i; \operatorname{rad} \mathbf{b}_i], \quad i = 1, 2, \ldots, m,$$

which is equivalent to

$$\left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \leqslant \operatorname{rad} \mathbf{b}_i, \quad i = 1, 2, \ldots, m.$$

Therefore, x actually belongs to $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ if and only if

$$\operatorname{Tol}(x; \mathbf{A}, \mathbf{b}) = \min_{1 \leqslant i \leqslant m} \left\{ \operatorname{rad} \mathbf{b}_i - \left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \right\} \geqslant 0. \qquad \square$$

Notice that the functional $\operatorname{Tol}(x; \mathbf{A}, \mathbf{b})$ is continuous (even Lipschitz-continuous) on all of its arguments.

Lemma 3. *The functional* $\operatorname{Tol}(x)$ *is concave.*

Proof. The functional $\operatorname{Tol}(x)$ is the lower envelope of the functionals

$$\xi_i(x) = \operatorname{rad} \mathbf{b}_i - \left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right|, \quad i = 1, 2, \ldots, m,$$

and we need only to establish the concavity of each $\xi_i(x)$.

Let $x, y \in \mathbb{R}^n$, $\lambda \in [0; 1]$. The subdistributivity of the interval arithmetic [1–3,9,11] then implies

$$\operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} (\lambda x_j + (1-\lambda) y_j) \subseteq \lambda \left(\operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right) + (1-\lambda) \left(\operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} y_j \right).$$

The magnitude $|\cdot|$ is isotonic with respect to the inclusion ordering of intervals and the standard linear order on $\mathbb{R}$ [11]. Hence,

$$\begin{aligned} \left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} (\lambda x_j + (1-\lambda) y_j) \right| &\leqslant \left| \lambda \left(\operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right) + (1-\lambda) \left(\operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} y_j \right) \right| \\ &\leqslant \lambda \left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| + (1-\lambda) \left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right|, \end{aligned}$$

and the assertion of the Lemma follows. $\square$

Thus, the ordinate set

$$\operatorname{hyp} \operatorname{Tol} = \{ (x, z) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n, z \in \mathbb{R}, \operatorname{Tol}(x) \leqslant z \}$$

of the map $\operatorname{Tol}: \mathbb{R}^n \to \mathbb{R}$ is a convex set. We shall show that hyp Tol is the intersection of a finite number of half-spaces of $\mathbb{R}^{n+1}$, i.e., it is a *convex polyhedral set* according to the terminology by Rockafellar [19]. Indeed, expressing the absolute value in terms of maximum, we get for each $i = 1, 2, \ldots, m$

$$\begin{aligned}
&\operatorname{rad} \mathbf{b}_i - \left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \\
&= \operatorname{rad} \mathbf{b}_i - \max_{\hat{a}_{ij}} \left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \hat{a}_{ij} x_j \right| \\
&= \operatorname{rad} \mathbf{b}_i - \max_{\hat{a}_{ij}} \left\{ \max \left\{ \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \hat{a}_{ij} x_j ,\ \sum_{j=1}^{n} \hat{a}_{ij} x_j - \operatorname{mid} \mathbf{b}_i \right\} \right\} \\
&= \min_{\hat{a}_{ij}} \left\{ \min \left\{ \operatorname{rad} \mathbf{b}_i - \operatorname{mid} \mathbf{b}_i + \sum_{j=1}^{n} \hat{a}_{ij} x_j ,\ \operatorname{rad} \mathbf{b}_i + \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \hat{a}_{ij} x_j \right\} \right\},
\end{aligned}$$

where the n-tuple $(\hat{a}_{i1}, \hat{a}_{i2}, \ldots, \hat{a}_{in})$ runs over the finite set vert $(\mathbf{a}_{i1}, \mathbf{a}_{i2}, \ldots, \mathbf{a}_{in})$, that is, over all vertices of the i-th row of the interval matrix $\mathbf{A}$. Owing to this, the functional Tol is the lower envelope of at most $m \cdot 2^{n+1}$ affine functionals of the form

$$\operatorname{rad} \mathbf{b}_i \pm \left(\operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \hat{a}_{ij} x_j \right),$$

$i = 1, 2, \ldots, m$, and the set hyp Tol is intersection of these functionals' ordinate sets.

As a consequence we get the following well known result: *tolerable solution set is a convex polyhedral set.*

Lemma 4. *The functional* $\operatorname{Tol}(x)$ *attains a finite maximum on all of* $\mathbb{R}^n$.

Proof. Being a convex polyhedral set, the ordinate set hyp Tol is the convex hull of a finite set of points (c_k, γ_k), $k = 1, 2, \ldots, p$, and directions (c_k, γ_k), $k = p+1, \ldots, q$, of $\mathbb{R}^{n+1}$ (excluding the direction $(0, \ldots, 0, 1)$ since Tol (x) is defined everywhere) [19]. More precisely,

$$\operatorname{hyp} \operatorname{Tol} = \left\{ \sum_{k=1}^{q} \lambda_k (c_k, \gamma_k) \,\middle|\, c_k \in \mathbb{R}^n,\ \gamma_k, \lambda_k \in \mathbb{R},\ \lambda_k \geqslant 0,\ \sum_{k=1}^{p} \lambda_k = 1 \right\}.$$

Inasmuch as Tol $(x) \leqslant \min_{1 \leqslant i \leqslant m} \operatorname{rad} \mathbf{b}_i$, we must conclude that $\gamma_k \leqslant 0$, $k = p+1, \ldots, q$, since otherwise the functional Tol would be unbounded from above. For this reason,

$$\begin{aligned}
\max_{x \in \mathbb{R}^n} \operatorname{Tol}(x) &= \max\{ z \mid (x, z) \in \operatorname{hyp} \operatorname{Tol},\ x \in \mathbb{R}^n,\ z \in \mathbb{R} \} \\
&= \max \left\{ \sum_{k=1}^{q} \lambda_k \gamma_k \,\middle|\, \lambda_k \geqslant 0,\ \sum_{k=1}^{p} \lambda_k = 1 \right\} \\
&= \max \left\{ \sum_{k=1}^{p} \lambda_k \gamma_k \,\middle|\, \lambda_k \geqslant 0,\ \sum_{k=1}^{p} \lambda_k = 1 \right\} \\
&= \max_{1 \leqslant k \leqslant p} \gamma_k .
\end{aligned}$$

The sought-for maximum thus coincides with that over a finite number of the functional's values, and $\max_{x \in \mathbb{R}^n} \text{Tol}\,(x)$ is attainable together with all γ_k, $k = 1, 2, \ldots, p$. □

Lemma 5. *If the interval matrix* $\mathbf{A}$ *does not have zero rows, then* $t \in \text{int}\,\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ *implies* $\text{Tol}\,(t; \mathbf{A}, \mathbf{b}) > 0$.

Proof. Let $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \neq \emptyset$ and $\max \text{Tol}\,(x)$ is reached at some point $\tau \in \Sigma_{\forall\exists}$. If $t \in \text{int}\,\Sigma_{\forall\exists}$, then t is an interior point of a segment $[\tau; y] \subset \Sigma_{\forall\exists}$, i.e. $t = \lambda\tau + (1-\lambda)y$ for some $\lambda \in (0; 1)$, $y \in \Sigma_{\forall\exists}$. Therefore

$$\text{Tol}(t) \geqslant \lambda\, \text{Tol}(\tau) + (1-\lambda)\text{Tol}(y),$$

because the functional Tol is concave.

Suppose $\text{Tol}(t) = 0$. Then the above inequality holds only when $\text{Tol}(\tau) = \text{Tol}(y) = 0$ and the functional Tol must equal zero on the entire set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$. Furthermore, let $\mathbb{R}^n = \bigcup_{1 \leqslant i \leqslant m} \mathscr{O}_i$ with

$$\mathscr{O}_i = \left\{ x \in \mathbb{R}^n \,\middle|\, \text{Tol}(x) = \text{rad}\,\mathbf{b}_i - \left| \text{mid}\,\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \right\}.$$

It is fairly simple to see that

$$\Sigma_{\forall\exists} = \bigcup_{1 \leqslant i \leqslant m} (\Sigma_{\forall\exists} \cap \mathscr{O}_i),$$

all the sets $\Sigma_{\forall\exists} \cap \mathscr{O}_i$, $i = 1, 2, \ldots, m$, being closed. Hence, $\text{int}\,(\Sigma_{\forall\exists} \cap \mathscr{O}_i) \neq \emptyset$ for at least one $k \in \{1, 2, \ldots, m\}$ and we have

$$\text{rad}\,\mathbf{b}_k - \left| \text{mid}\,\mathbf{b}_k - \sum_{j=1}^{n} \mathbf{a}_{kj} x_j \right| = 0 = \text{const}$$

for all $x \in \text{int}(\Sigma_{\forall\exists} \cap \mathscr{O}_k)$. The latter may occur only when all $\mathbf{a}_{k1}, \ldots, \mathbf{a}_{kn}$ are zeros, which contradicts the assertion of the Lemma. □

Lemma 6. *If* $\text{Tol}\,(t; \mathbf{A}, \mathbf{b}) > 0$, *then* $t \in \text{int}\,\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \neq \emptyset$.

Proof. The map Tol: $\mathbb{R}^n \to \mathbb{R}$ is continuous, so the set $Y = \{y \in \mathbb{R}^n \mid \text{Tol}(y) > 0\}$ is open. Also, it is nonempty — $t \in Y \subseteq \Sigma_{\forall\exists}$ — and $Y \subseteq \text{int}\,\Sigma_{\forall\exists}$. Hence, $x \in \text{int}\,\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \neq \emptyset$. □

To summarize, we come to the following technique to investigate solvability of the linear tolerance problem, i.e., to the criterion for the tolerable solution set to be nonempty:

Solve the unconstrained maximization problem for the concave functional

$$\text{Tol}(x; \mathbf{A}, \mathbf{b}) = \min_{1 \leqslant i \leqslant m} \left\{ \text{rad}\,\mathbf{b}_i - \left| \text{mid}\,\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \right\}.$$

Let $T = \max_{x \in \mathbb{R}^n} \text{Tol}(x; \mathbf{A}, \mathbf{b})$ and let T be reached at a point τ. We have

- if $T \geqslant 0$, then $\tau \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \neq \emptyset$, i.e., the linear tolerance problem is compatible, and if $T > 0$, then $\tau \in \text{int}\,\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \neq \emptyset$;
- if $T < 0$, then $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \emptyset$, i.e., the linear tolerance problem is incompatible.

It is worth noting that Lemmae 1–4 as well as the above solvability criterion would remain valid if the functional Tol was defined by the expression

$$\min_{1 \leqslant i \leqslant m} \left\{ \zeta_i \left(\text{rad } \mathbf{b}_i - \left| \text{mid } \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \right) \right\},$$

where ζ_i, $i = 1, 2, \dots, m$, are positive reals. Below are some examples in which such functionals naturally come into existence and then are employed fruitfully.

Maximization of nonsmooth concave functions has been much studied during the last few decades. A good many efficient numerical methods have been proposed to solve this problem (see [7,31] et al) and this is reason to hope that the solvability criterion developed above is quite practical.

5. Correction of the linear tolerance problem

Imagine solving an actual practical problem. Usually, the effort does not terminate even after we reach the conclusion that the problem has no solutions (unsolvable). A client is very likely to be interested in an information about

- how unsolvable the problem is,
- how one must change the input data to make the problem solvable, and so on.

Alternately, if the original problem proves to be solvable, then, frequently, the region of variations of input data within which the problem remains solvable is to be outlined. We are able to give quite expanded answers to some of these questions.

If **A** and mid **b** are unchanged, increasing the radii of all the components of **b** by the same value K is easily seen to lead to adding the constant K to the functional Tol (x). Therefore,

$$\max_{x \in \mathbb{R}^n} \text{Tol}(x; \mathbf{A}, \mathbf{b} + K\mathbf{e}) = K + \max_{x \in \mathbb{R}^n} \text{Tol}(x; \mathbf{A}, \mathbf{b}),$$

where $\mathbf{e} = ([-1; 1], \dots, [-1; 1])^{\mathrm{T}}$. If the linear tolerance problem is unsolvable and

$$\max_{x \in \mathbb{R}^n} \text{Tol}(x; \mathbf{A}, \mathbf{b}) = T < 0,$$

we can make it solvable with the same matrix **A** through widening the right-hand side vector by $K\mathbf{e}$, $K \geqslant 0$, and the points $\tau \in \text{Arg max Tol}(x; \mathbf{A}, \mathbf{b})$ will certainly belong to the nonempty set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b} + K\mathbf{e})$. Conversely, if

$$\max_{x \in \mathbb{R}^n} \text{Tol}(x; \mathbf{A}, \mathbf{b}) = T \geqslant 0,$$

that is, the linear tolerance problem is solvable, it will remain so even after we decrease the radii of all right-hand side components by K, $K \leqslant T$.

Sometimes, such uniform widening of all the components of **b** may prove unacceptable in practice. So, let us assume that a vector $\upsilon = (\upsilon_1, \upsilon_2, \dots, \upsilon_m)$, $\upsilon_i > 0$, is given such that the increase of the width of $\mathbf{b}_i$ is to be proportional to υ_i. Now, calculate

$$T_\upsilon = \max_{x \in \mathbb{R}^n} \text{Tol}_\upsilon(x; \mathbf{A}, \mathbf{b}),$$

where

$$\mathrm{Tol}_v(x) = \min_{1 \leqslant i \leqslant m} \left\{ v_i^{-1} \left(\mathrm{rad}\, \mathbf{b}_i - \left| \mathrm{mid}\, \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \right) \right\}. \tag{17}$$

If, for instance, initially, the linear tolerance problem with the matrix **A** and the right-hand side vector **b** had no solutions, then the problem with the same matrix **A** and the expanded vector $(\mathbf{b}_i + K v_i [-1; 1])_{i=1}^m$ in the right-hand side becomes solvable for $K \geqslant |T_v|$.

The most important particular case of the above construction is that of ensuring equal relative (proportional to the absolute values) increases of the radii of the right-hand side components, when $v_i = |\mathbf{b}_i|$ for nonzero $\mathbf{b}_i$, $i = 1, 2, \dots, m$. Denote

$$\mathrm{Tol}_0(x) = \min_{1 \leqslant i \leqslant m} \left\{ |\mathbf{b}_i|^{-1} \left(\mathrm{rad}\, \mathbf{b}_i - \left| \mathrm{mid}\, \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \right) \right\}$$

and let

$$T_0 = \max_{x \in \mathbb{R}^n} \mathrm{Tol}_0(x; \mathbf{A}, \mathbf{b}).$$

The magnitude of T_0 is the far more subtle quantitative characteristic of compatibility of the linear tolerance problem than

$$\Omega = \min_{1 \leqslant i \leqslant m} \left\{ \max_{1 \leqslant j \leqslant n} \chi(\mathbf{a}_{ij}) - \chi(\mathbf{b}_i) \right\}$$

introduced in Section 2. Judging by the absolute value of T_0, one can precisely estimate the degree of unsolvability in the case $T_0 < 0$ and the reserve of solvability (stability of the solvable state) in the case $T_0 \geqslant 0$. Naturally, all this is attained at the price of more laborious computation.

We have demonstrated some capabilities to correct the linear tolerance problem by modification of only the right-hand side vector **b**. In fact, the tolerance problem can also be corrected through varying the elements of the matrix **A** as well, and the basis of the corresponding technique is

Lemma 7. *Let* **x** *be an interval and* **s** *be a symmetric interval such that* rad **x** $\geqslant$ rad **s**. *Then* $\mathbf{x} \ominus \mathbf{s}$ *is also an interval and* $|\mathbf{x} \ominus \mathbf{s}| = |\mathbf{x}| - |\mathbf{s}|$.

Proof. Denote for brevity $s := |\mathbf{s}| = \mathrm{rad}\, \mathbf{s}$. Also, assume for definiteness that

$$|\mathbf{x}| = \max\{|\overline{\mathbf{x}}|, |\underline{\mathbf{x}}|\} = |\overline{\mathbf{x}}|,$$

i.e., $|\overline{\mathbf{x}}| \geqslant |\underline{\mathbf{x}}|$. In particular, this means $\overline{\mathbf{x}} \geqslant 0$.

It turns out that under our assumption

$$|\overline{\mathbf{x}} - s| \geqslant |\underline{\mathbf{x}} + s|. \tag{18}$$

Indeed, the statement of the Lemma implies

$$\mathrm{rad}\, \mathbf{x} = \frac{\overline{\mathbf{x}} - \underline{\mathbf{x}}}{2} \geqslant s,$$

that is,

$$\overline{\mathbf{x}} - s \geqslant \underline{\mathbf{x}} + s.$$

If $\underline{\mathbf{x}} + s \geqslant 0$, then the inequality (18) is obtained from that by taking the absolute values of both sides. Otherwise, if $\underline{\mathbf{x}} + s \leqslant 0$, then $\underline{\mathbf{x}} \leqslant 0$, $|\underline{\mathbf{x}}| = -\underline{\mathbf{x}}$ and we again have

$$|\overline{\mathbf{x}} - s| = \overline{\mathbf{x}} - s \geqslant |\underline{\mathbf{x}}| - s = -\underline{\mathbf{x}} - s = |\underline{\mathbf{x}} + s|.$$

Finally, in view of (18)

$$|\mathbf{x} \ominus \mathbf{s}| = \max\{|\overline{\mathbf{x}} - s|, |\underline{\mathbf{x}} + s|\} = |\overline{\mathbf{x}} - s| = \overline{\mathbf{x}} - s = |\mathbf{x}| - |\mathbf{s}|$$

as required. For $|\mathbf{x}| = |\underline{\mathbf{x}}|$, the proof is conducted in the similar way. □

Assume that we are given an incompatible linear tolerance problem with the interval matrix $\mathbf{A}$ and the interval right-hand vector $\mathbf{b}$. Accordingly the unconstrained maximum of its identifying functional Tol $(x;\ \mathbf{A}, \mathbf{b})$, which we suppose to be attained at the point τ, is negative, i.e.,

$$\max_{x \in \mathbb{R}^n} \text{Tol}(x;\ \mathbf{A}, \mathbf{b}) = \text{Tol}(\tau;\ \mathbf{A}, \mathbf{b}) = T < 0.$$

How can one diminish unsolvability measure of the linear tolerance problem through narrowing the matrix $\mathbf{A}$?

In doing so, we shall take the following natural assumptions:

(i) all the components of the right-hand side vector $\mathbf{b}$ are thick intervals, that is, rad $\mathbf{b}_i > 0$, $i = 1, 2, \ldots, m$,

(ii) $\min_{1 \leqslant i \leqslant m}\{\sum_{j=1}^{n} \tau_j \text{ rad } \mathbf{a}_{ij}\} = \Delta > 0$.

Let us choose an interval $m \times n$-matrix $\mathbf{E} = (\mathbf{e}_{ij})$ with symmetric interval elements $\mathbf{e}_{ij} = [-e_{ij};\ e_{ij}]$ so that

$$\sum_{j=1}^{n} e_{ij}\tau_j = K, \quad i = 1, 2, \ldots, m, \tag{19}$$

where K is a positive constant $\leqslant \Delta$ and, of course,

$$\text{rad } \mathbf{a}_{ij} \geqslant e_{ij} \geqslant 0 \tag{20}$$

for all i, ij. Then the linear tolerance problem with that same right-hand side vector $\mathbf{b}$ and the interval matrix $\mathbf{A} \ominus \mathbf{E}$ is "less unsolvable" than the initial one.

Indeed, evaluating the identifying functional of the new tolerance problem, we have

$$\begin{aligned} \text{Tol}(x;\ \mathbf{A} \ominus \mathbf{E}, \mathbf{b}) &= \min_{1 \leqslant i \leqslant m} \left\{ \text{rad } \mathbf{b}_i - \left| \text{mid } \mathbf{b}_i - \sum_{j=1}^{n} (\mathbf{a}_{ij} \ominus \mathbf{e}_{ij}) x_j \right| \right\} \\ &= \min_{1 \leqslant i \leqslant m} \left\{ \text{rad } \mathbf{b}_i - \left| \text{mid } \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \ominus \sum_{j=1}^{n} \mathbf{e}_{ij} x_j \right| \right\} \\ &= \min_{1 \leqslant i \leqslant m} \left\{ \text{rad } \mathbf{b}_i - \left| \text{mid } \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| + \left| \sum_{j=1}^{n} \mathbf{e}_{ij} x_j \right| \right\} \end{aligned}$$

by Lemma 7. Hence, since (19) is equivalent to

$$\left|\sum_{j=1}^{n} \mathbf{e}_{ij}\tau_j\right| = K,$$

we get

$$\begin{aligned}
\max_{x\in\mathbb{R}^n} \operatorname{Tol}(x; \mathbf{A}\ominus\mathbf{E}, \mathbf{b}) &\geqslant \operatorname{Tol}(\tau; \mathbf{A}\ominus\mathbf{E}, \mathbf{b}) \\
&= \min_{1\leqslant i\leqslant m} \left\{\operatorname{rad}\mathbf{b}_i - \left|\operatorname{mid}\mathbf{b}_i - \sum_{j=1}^{n}\mathbf{a}_{ij}\tau_j\right| + \left|\sum_{j=1}^{n}\mathbf{e}_{ij}\tau_j\right|\right\} \\
&= \min_{1\leqslant i\leqslant m} \left\{\operatorname{rad}\mathbf{b}_i - \left|\operatorname{mid}\mathbf{b}_i - \sum_{j=1}^{n}\mathbf{a}_{ij}x_j\right| + K\right\} \\
&= K + \operatorname{Tol}(\tau; \mathbf{A}, \mathbf{b}) \\
&= K + \max_{x\in\mathbb{R}^n} \operatorname{Tol}(x; \mathbf{A}, \mathbf{b}) \\
&= K + T.
\end{aligned}$$

If $K \geqslant |T|$ then the linear tolerance problem with the matrix $\mathbf{A}\ominus\mathbf{E}$ and the right-hand side $\mathbf{b}$ becomes compatible, and moreover, we may assert for sure that $\tau \in \Sigma_{\forall\exists}(\mathbf{A}\ominus\mathbf{E}, \mathbf{b})$.

The crucial point of the matrix variation correction procedure is the solution of the underdetermined system of equations (19)–(20). As one can see, sometimes the correction obtained through our prescription may turn out insufficient to make the tolerance problem certainly solvable (if $\Delta \leqslant |T|$), but one should perceive that rather as a drawback of our technique or as a consequence of the estimate's roughness. In principle, every linear tolerance problem with a nonsingular interval matrix $\mathbf{A}$ can be made compatible by appropriate narrowing the matrix, since for nonsingular thin $\mathbf{A}$ the problem always has a solution. To turn the initial problem into a compatible one for large $|T|$, we thereby recommend to apply the proposed expedient repeatedly, combined with recalculations of the identifying functional's maximum.

In the above consideration, we decreased the weighted (with the coefficients τ_j) width of each row of the interval matrix $\mathbf{A}$ by the same value K. Similar to the preceding case, one may need to decrease those widths to a variable degree. The way out of the situation is standard: we introduce the positive vector $v = (v_1, v_2, \ldots, v_m)$ such that the decrease measure (19) of the i-th row's weighted width should be proportional to v_i and then operate with the modified functional $\operatorname{Tol}_v(x; \mathbf{A}, \mathbf{b})$ which is defined by the expression (17).

6. Formula for the size of an interval solution

Once the compatibility of the linear tolerance problems is established and a point of the tolerable solution set has been found, we may turn to the actual construction of the interval solution to the problem. To do so, we follow the so-called "center" approach adopted by Khlebalin [5], Neumaier [11], Shaidurov [2], Shaidurov and Shary [23] and others, in which the point of the tolerable solution set found earlier is taken to be the center of the interval solution under construction. Taking the minimum on a hyperbar plays a leading part in the formula derived below, so that the further solution of the linear tolerance problem amounts to a finite-dimensional constrained optimization problem on a hyperbar. In what follows, we do not discuss optimal choice of the interval solution's center since that question is closely connected with the practical needs of those who solve specific problems.

In applications, the statement of the linear tolerance problem is often more rigid than (6). Like Shaidurov in [2,23], in addition to (6), we take the ratio of the tolerances of the separate components to be determined by a real vector $w = (w_1, w_2, \dots, w_n)$, $w_i > 0$, i.e., we introduce weighting coefficients for the widths of the tolerances so that

$$\text{rad } \mathbf{U}_i / \text{rad } \mathbf{U}_j = w_i / w_j.$$

Through scaling by the diagonal matrix $\text{diag}\{w_1, w_2, \dots, w_n\}$ all such cases are easily reduced to a standard one, when $w = (1, 1, \dots, 1)$ and we are to inscribe a hypercube in the properly modified set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$.

Indeed, we introduce matrices $D = \text{diag}\{w_1, w_2, \dots, w_n\}$ and $\tilde{\mathbf{A}} = \mathbf{A}D$. Let the interval vector $\tilde{\mathbf{U}}$, rad $\tilde{\mathbf{U}}_i$ = const, be a solution to the linear tolerance problem with the matrix $\tilde{\mathbf{A}}$ and the right-hand side vector $\mathbf{b}$. Then $\mathbf{U} = D\tilde{\mathbf{U}}$ is a solution to the original problem, since

$$\{\mathbf{A}x \mid x \in \mathbf{U}\} = \{\mathbf{A}DD^{-1}x \mid x \in \mathbf{U}\} = \left\{\tilde{\mathbf{A}}\tilde{x} \mid \tilde{x} \in \tilde{\mathbf{U}}\right\} \subseteq \mathbf{b},$$

and moreover $\text{rad } \mathbf{U}_i / \text{rad } \mathbf{U}_j = w_i / w_j$ as required. That is why from now on the linear tolerance problem will be referred to as a problem of finding an interval vector $\mathbf{U}$ with components of equal width and such that $\{\mathbf{A}x \mid x \in \mathbf{U}\} \subseteq \mathbf{b}$.

The basis of all our further account is the following

Theorem 2. *If* $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ *then for*

$$r = \min_{1 \le i \le m} \min_{A \in \text{vert } \mathbf{A}} \left\{ \frac{\text{rad } \mathbf{b}_i - |\text{mid } \mathbf{b}_i - \sum_{j=1}^n a_{ij} t_j|}{\sum_{j=1}^n |a_{ij}|} \right\} \tag{21}$$

the interval vector $\mathbf{U} = (t + r\mathbf{e})$ *is also entirely contained in* $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$.

Proof. First assume that in the linear tolerance problem the matrix $\mathbf{A}$ is thin, i.e., $\mathbf{A} = A$ and vert $\mathbf{A} = A$. We represent each $x \in \mathbf{U}$ in the form $x = t + y$, where $\max_{1 \le j \le n} |y_j| \le r_A$ and

$$r_A = \min_{1 \le i \le m} \left\{ \frac{\text{rad } \mathbf{b}_i - |\text{mid } \mathbf{b}_i - \sum_{j=1}^n a_{ij} t_j|}{\sum_{j=1}^n |a_{ij}|} \right\}, \tag{22}$$

so that the following holds for $i = 1, 2, \dots, m$:

$$|(Ay)_i| = \left| \sum_{j=1}^n a_{ij} y_j \right| \le \sum_{j=1}^n |a_{ij}| \, |y_j| \le r_A \cdot \sum_{j=1}^n |a_{ij}| \le \text{rad } \mathbf{b}_i - \left| \text{mid } \mathbf{b}_i - \sum_{j=1}^n a_{ij} t_j \right|.$$

Therefore, since $Ax = At + Ay$, we obtain

$$(At)_i - \text{rad } \mathbf{b}_i + |\text{mid } \mathbf{b}_i - (At)_i| \le (Ax)_i \le (At)_i + \text{rad } \mathbf{b}_i - |\text{mid } \mathbf{b}_i - (At)_i|$$

or, equivalently,

$$\begin{aligned} &\underline{\mathbf{b}}_i - (\operatorname{mid} \mathbf{b}_i - (At)_i) + |\operatorname{mid} \mathbf{b}_i - (At)_i| \\ &\quad \leqslant (Ax)_i \\ &\quad \leqslant \overline{\mathbf{b}}_i - (\operatorname{mid} \mathbf{b}_i - (At)_i) - |\operatorname{mid} \mathbf{b}_i - (At)_i|. \end{aligned} \tag{23}$$

By virtue of the fact that

$$-z + |z| \geqslant 0 \quad \text{and} \quad -z - |z| \leqslant 0$$

for any real z, the inequality (23) implies

$$\underline{\mathbf{b}}_i \leqslant (Ax)_i \leqslant \overline{\mathbf{b}}_i,$$

that is, $Ax \in \mathbf{b}$ as was expected.

Now, let the matrix $\mathbf{A}$ of the problem be a thick interval matrix, and $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \neq \emptyset$. We consider the totality of all linear tolerance problems for systems $Ax = \mathbf{b}$ with $A \in \mathbf{A}$. According to the representation (11),

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \bigcap_{A \in \operatorname{vert} \mathbf{A}} \Sigma_{\forall\exists}(A, \mathbf{b}),$$

and if for each $A \in \operatorname{vert} \mathbf{A}$ the corresponding interval solution vector is $\mathbf{U}_A$, $\mathbf{U}_A \subseteq \Sigma_{\forall\exists}(A, \mathbf{b})$, then the interval vector $\mathbf{U}$ such that

$$\mathbf{U} = \bigcap_{A \in \operatorname{vert} \mathbf{A}} \mathbf{U}_A$$

is also included in $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$. In particular, when all $\mathbf{U}_A$ have a common center and their radii are defined by formula (22), we have

$$\mathbf{U} = t + r\mathbf{e},$$

where

$$r = \min_{A \in \operatorname{vert} \mathbf{A}} r_A = \min_{1 \leqslant i \leqslant m} \min_{A \in \operatorname{vert} \mathbf{A}} \left\{ \frac{\operatorname{rad} \mathbf{b}_i - |\operatorname{mid} \mathbf{b}_i - \sum_{j=1}^n a_{ij} t_j|}{\sum_{j=1}^n |a_{ij}|} \right\}.$$

The Theorem is completely proved. □

Despite outward simplicity of that proof, the statement of Theorem 6 is the most subtle of that kind of results. Previously, Shaidurov has established in [2,23] that if $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ then for

$$r = \min_{1 \leqslant i \leqslant m} \min_{A \in \mathbf{A}} \left\{ \frac{\operatorname{rad} \mathbf{b}_i |\operatorname{mid} \mathbf{b}_i - \sum_{j=1}^n a_{ij} t_j|}{\sum_{j=1}^n |a_{ij}|} \right\} \tag{24}$$

the interval vector $(t + r\mathbf{e})$ is included in $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$. As is seen, his formula coincides with ours except for the internal minimum is taken over *all* matrices $A \in \mathbf{A}$ and not over the *finite* set of extreme matrices. That, nevertheless, in no way affects the final results which are completely identical for both formulas. Indeed, one can easily show quasiconcavity of the functions in the braces of (24) (that was done by Shary in [25,28]), and a quasiconcave function is known to reach its minimum in extreme points of its convex domain of definition. Thus, for each

$i = 1, 2, \ldots, m$, the expressions in braces of (24) attain their minimal values on $A \in \mathbf{A}$ in vertices of the interval vectors $(\mathbf{a}_{i1}, \mathbf{a}_{i2}, \ldots, \mathbf{a}_{in})$ and the formula (21) follows. In this work, we achieve the same objective by the more elementary means.

The simplest way to estimate (24) and (21) is to take the left endpoint of the *natural interval extension* over $\mathbf{A}$ [9] for the expressions in the braces of (24), that is, to replace the variables by the corresponding intervals of their domains and to replace the arithmetic operations by the corresponding interval arithmetic operations. The following algorithm by Shaidurov [2,23] does exactly so.

Algorithm 1.
For a given $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$, calculate the intervals

$$\mathbf{r}_i = \frac{\text{rad}\,\mathbf{b}_i - |\text{mid}\,\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} t_j|}{\sum_{j=1}^{n} |\mathbf{a}_{ij}|}, \tag{25}$$

$i = 1, 2, \ldots, m$, and then put $\rho := \min_{1 \leq i \leq m} \underline{\mathbf{r}}_i$. The interval vector $(t + \rho\mathbf{e})$ is a solution to the linear tolerance problem.

Since both numerator and denominator of the minimized expression contain only one occurrence of each variable in the first power, Shaidurov's algorithm is actually equivalent to estimation of a fraction as the quotient of the numerator's minimum and denominator's maximum. The relative accuracy of such estimation is proved in [2,23] to be higher, the more narrow the matrix $\mathbf{A}$.

The other important result on construction of an interval solution to the linear tolerance problem is due to Neumaier [10], who has proposed the following simple method.

Algorithm 2.
For a given $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$, calculate the largest nonnegative η such that

$$\eta \cdot \mathbf{A}\mathbf{e} \subseteq \mathbf{b} \ominus \mathbf{A}t. \tag{26}$$

The interval vector $(t + \eta\mathbf{e})$ is a solution to the linear tolerance problem.

The latter is evident from the fact that

$$\mathbf{A}x \subseteq \mathbf{A}(t + \eta\mathbf{e}) \subseteq \mathbf{A}t + \mathbf{A}(\eta\mathbf{e}) \subseteq \mathbf{A}t + \mathbf{b} \ominus \mathbf{A}t = \mathbf{b}$$

for each $x \in t + \eta\mathbf{e}$.

Lemma 8. *The results obtained by Neumaier's algorithm are completely identical to those given by Shaidurov's algorithm.*

Proof. In fact, condition (26) means

$$\eta \cdot \underline{(\mathbf{A}\mathbf{e})}_i \geq \underline{(\mathbf{b} \ominus \mathbf{A}t)}_i \quad \text{and} \quad \eta \cdot \overline{(\mathbf{A}\mathbf{e})}_i \leq \overline{(\mathbf{b} \ominus \mathbf{A}t)}_i, \quad i = 1, 2, \ldots, m,$$

where **Ae** is a symmetrical interval vector in which

$$-(\underline{\mathbf{Ae}})_i = (\overline{\mathbf{Ae}})_i = |(\mathbf{Ae})_i|, \quad i = 1, 2, \dots, m.$$

Moreover, for $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$

$$(\underline{\mathbf{b} \ominus \mathbf{A}t})_i \leqslant 0 \leqslant (\overline{\mathbf{b} \ominus \mathbf{A}t})_i$$

and thus the following chain of transformations is valid for each i:

$$\eta \leqslant \min\left\{ \frac{(\underline{\mathbf{b} \ominus \mathbf{A}t})_i}{(\underline{\mathbf{Ae}})_i}, \frac{(\overline{\mathbf{b} \ominus \mathbf{A}t})_i}{(\overline{\mathbf{Ae}})_i} \right\}$$

$$= \frac{\min\left\{ -(\text{mid } \mathbf{b}_i - \text{rad } \mathbf{b}_i) + (\underline{\mathbf{A}t})_i, (\text{mid } \mathbf{b}_i + \text{rad } \mathbf{b}_i) - (\overline{\mathbf{A}t})_i \right\}}{|(\mathbf{Ae})_i|}$$

$$= \frac{\min\left\{ \text{rad } \mathbf{b}_i - \left(\text{mid } \mathbf{b}_i - (\underline{\mathbf{A}t})_i\right), \text{rad } \mathbf{b}_i - \left((\overline{\mathbf{A}t})_i - \text{mid } \mathbf{b}_i\right) \right\}}{|(\mathbf{Ae})_i|}$$

$$= \frac{\text{rad } \mathbf{b}_i - \max\left\{ \text{mid } \mathbf{b}_i - (\underline{\mathbf{A}t})_i, (\overline{\mathbf{A}t})_i - \text{mid } \mathbf{b}_i \right\}}{|(\mathbf{Ae})_i|}$$

$$= \frac{\text{rad } \mathbf{b}_i - |\text{mid } \mathbf{b}_i - (\mathbf{A}t)_i|}{|(\mathbf{Ae})_i|}.$$

For $i = 1, 2, \dots, m$, the last expression coincides with the lower bounds of the respective interval (25), and so taking the minimum on all i leads to the equality $\eta = \varrho$. □

Both these algorithms, by Shaidurov and by Neumaier, are simple and easy to implement: if a point $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ has been already found, then the construction of an interval solution requires as low as $O(mn)$ arithmetical operations. This is achieved, however, at the price of considerable coarsening of the final result, especially for wide interval **A**.

In the expression (21), taking the minimum on $i \in \{1, 2, \dots, m\}$ involves no difficulties, so the central problem is computation of the internal $\min_{A \in \text{vert } \mathbf{A}}$ or its estimate from below. As a matter of fact, the rest of the paper will be devoted to solving this problem.

For the sake of convenience, we denote

$$(\mathbf{a}_{i1}, \mathbf{a}_{i2}, \dots, \mathbf{a}_{in}) = (\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_n) = \mathbf{D}$$

regardless of the index i, while the objective function will be the expression in braces (21) and (24), that is,

$$\Phi(x) = \frac{R - |M - \sum_{j=1}^{n} x_j t_j|}{\sum_{j=1}^{n} |x_j|}.$$

Thus, the problem to be solved is

$$\text{Find} \quad \min\{\Phi(x) \mid x \in \text{vert } \mathbf{D}\}.$$

7. Exact exhaustive algorithms to construct an interval solution

When algoritmically implementing the exhaustion of vert **D**, one usually indexes the vertices with the n-digit binary numbers from $00\cdots 0$ to $11\cdots 1$, the k-th digit being equal to 0 if the k-th coordinate of the respective vertex coincides with the left endpoint of the interval $\mathbf{d}_k$, and 1 if it coincides with the right one. Then item-by-item examination of the vertices of $(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n)$ can be organized, for instance, as a successive passage, starting from the vertex with the number 0 and further on, each time to the vertex that has the next binary number. If the steps of the exhaustion are also numbered, then the process we have considered is formally described in the following form.

Algorithm 3.
We examine the vertex with the binary number ν at the ν-th step.

We shall show how to considerably decrease the complexity of this algorithm (which is proportional to 2^n). At each step of the exhaustion, we have to calculate the sums

$$\Upsilon(d) = M - \sum_{j=1}^{n} d_j t_j \quad \text{and} \quad \Theta(d) = \sum_{j=1}^{n} |d_j|$$

for the examined vertex $(d_1, d_2, \ldots, d_n)$, so that $\Phi(d) = (R - |\Upsilon(d)|)/\Theta(d)$. If the item-by-item examination of the vertices is carried out in a special manner, passing at each step to an adjacent vertex (that has only one different component), then only one summand will change in each of the expressions Υ and Θ. So, to compute their new values, one does not need to sum all n terms anew. It is much more saving to recalculate Υ and Θ through the following recurrent formulae: if the new examined vertex is different from the preceding one in its j-th coordinate only – it is equal to d_j'' instead of d_j' – then

$$\text{the new value of } \Upsilon = \text{the old value of } \Upsilon + (d_j'' - d_j') \cdot t_j,$$

$$\text{the new value of } \Theta = \text{the old value of } \Theta + |d_j''| - |d_j'|.$$

Algorithm 3 does not possess the desired property of "passing to an adjacent vertex". For example, the subsequent binary numbers 011 and 100 differ in three digits and not in one as it should be for the neighboring vertices. So, to put the above idea to practice, one needs a special algorithm, which is actually a renumeration of the rectangle's vertices. We shall describe it recurrently, indexing the algorithm's steps by n-digit binary numbers as before.

Algorithm 4.
We examine the vertex with the number $00\cdots 00$ at the first step of the algorithm (that has the number $k = 0$). Let the number of the vertex μ_k be already determined. We denote the most significant digit's index in which the binary numbers k and $k+1$ differ by j. As the number μ_{k+1} of the vertex examined at the $(k+1)$-th step, we put the binary number which is obtained from μ_k through replacing its j-th digit by the complimentary one (that is, 0 by 1 and 1 by 0).

Lemma 9. *Algorithm 4 passes just once through each vertex of the rectangle* **D**, *and any two vertices with the numbers μ_k and μ_{k+1} are adjacent.*

Proof. The property of Algorithm 4 to perform the complete exhaustion of the vertices also remains valid if *any* other vertex is taken as the initial one and not only $00\cdots00$. We shall prove this more general fact by the induction on the dimension n of the rectangle.

Indeed, if $n=1$ the statement is evident for both binary sequences $\{0, 1\}$ and $\{1, 0\}$. Suppose it is already substantiated for the dimension $(n-1)$. We consider the renumeration of the vertices produced by Algorithm 4 for the dimension n:

$$\mu_{00\cdots00},\ \mu_{00\cdots01}, \ldots, \mu_{11\cdots11}. \tag{27}$$

It is significant that in this sequence the n-th digit changes only once, namely, when the step number passes from $01\ldots11$ to $10\ldots00$. So, no one of the numbers from the first half of the sequence (27) may be equal to a number from its second half because of the different n-th digits. Dropping the n-th digit from the numbers in (27), we get the two sequences that renumerate the vertices of the $(n-1)$-dimensional rectangle:

$$\mu'_{0\cdots00},\ \mu'_{0\cdots01}, \ldots, \mu'_{1\cdots11},$$
$$\mu''_{0\cdots00},\ \mu''_{0\cdots01}, \ldots, \mu''_{1\cdots11}.$$

They are produced by Algorithm IV, but with the different initial numbers $\mu'_{00\cdots0}$ and $\mu''_{00\cdots0}$, while $\mu''_{00\cdots0}=\mu'_{11\ldots1}$. By the induction assumption, each such sequence contains only different $(n-1)$-digit binary numbers from $00\cdots0$ to $11\cdots1$. Therefore, all the numbers from the sequence (27) are different from each other too. They are 2^n altogether and so each one of them occurs only once in (27).

The sequence (27) is thus an enumeration of vertices of the n-dimensional rectangle **D**. Besides, it is straightforwardly seen from the very description of Algorithm 4 that the vertices with the numbers ν and $\nu+1$ are adjacent. The Lemma is completely proved. □

In Algorithm 4, the reduction in complexity is large, the larger dimension of the problem, but the exponentiality is still not overcome. For this reason the practical significance of the exhaustive algorithms described in this section are limited only to the problems of moderate dimension.

8. Algorithms based on the "branch-and-bound" strategy

As is seen, we need a more advanced algorithm having precision better than that of Algorithms 1–2, but with complexity less than that of the exhaustive Algorithms 3–4 rested on Theorem 6. The algorithms presented below, with the well known "branch and bound method" as a basis, occupies an intermediate position between the simplest Algorithms 1–2 by Shaidurov–Neumaier and the exhaustive algorithms of the previous section. Its running time is exponential with respect to the dimension only in the worst case (as in all methods of this kind), but, due to the flexible computational scheme, it can be successfully applied to the problems of any size, though the precision to which the value (21) is found will be limited by the computer resources available.

It is common knowledge that a natural interval extension of the rational function $F(x_1, x_2, \ldots, x_n)$ in which each variable occurs only once and to the first power only gives the exact range of values provided that no division by a zero-containing interval takes place [1–3,9,11,17]. Thus, for $\mathbf{D} \in I\mathbb{R}^n$, $\mathbf{D} = (\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_n) = [\underline{d}_1;\ \bar{d}_1] \times [\underline{d}_2;\ \bar{d}_2] \times \cdots \times [\underline{d}_n;\ \bar{d}_n]$, the left point of the natural interval extension $\underline{F(\mathbf{D})}$ is the global minimum of F over $\mathbf{D}$, and its right endpoint $\overline{F(D)}$ is the global maximum of F over $\mathbf{D}$. The point is that we can find not only the values $\min\{F(x) \mid x \in \mathbf{D}\}$ and $\max\{F(x) \mid x \in \mathbf{D}\}$, but also the arguments of F that provide them, that is, the sets $\operatorname{Arg}\min\{F(x) \mid x \in \mathbf{D}\}$ and $\operatorname{Arg}\max\{F(x) \mid x \in \mathbf{D}\}$. Henceforth we shall designate them as $\operatorname{Arg} \underline{F(\mathbf{D})}$ and $\operatorname{Arg} \overline{F(\mathbf{D})}$ for brevity.

Indeed, if each variable has the only occurrence in the first power in the rational expression F, then the dependence of F upon x_i, for example, looks as follows:

$$\text{either} \quad F(x_i) = \alpha x_i + \beta \quad \text{or} \quad F(x_i) = \frac{1}{\alpha x_i + \beta},$$

where α, β are constants independent of x_i. In any event, $F(x_i)$ is a monotonic function of x_i (as for the second opportunity, this is true for $0 \notin \alpha \cdot [\underline{d}_i;\ \bar{d}_i] + \beta$), and so, for fixed $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$, the values $\min F(x_i)$ and $\max F(x_i)$ are attained at the endpoints of $[\underline{d}_i;\ \bar{d}_i]$ or, if $\alpha = 0$, at any point of $[\underline{d}_i;\ \bar{d}_i]$. Since such a reasoning holds for any variable x_i no matter what the values of the other variables are, then the sets $\operatorname{Arg} \underline{F(\mathbf{D})}$ and $\operatorname{Arg} \overline{F(\mathbf{D})}$ are either vertices of the rectangle $\mathbf{D}$ or its entire faces. How can one find them?

When executing any one of the four arithmetical operations with intervals, that is, addition, subtraction, multiplication or division, we can find out, simultaneously with the resulting interval, which endpoints of the initial intervals give, adding (subtracting, multiplicating or dividing), one or the other endpoint of the interval result. In subtracting, for instance, the maximum of the difference, i.e., the right endpoint of the resulting interval, is reached when the minuend is equal to the right endpoint and the subtrahend is equal to the left endpoint of the respective intervals.

To compute the product $[\underline{d}_1;\ \bar{d}_1] \cdot [\underline{d}_2;\ \bar{d}_2]$, one has to perform four multiplications, that is, to find $\{\bar{d}_1\bar{d}_2,\ \underline{d}_1\bar{d}_2,\ \bar{d}_1\underline{d}_2,\ \underline{d}_1\underline{d}_2\}$ and to select minimum and maximum from these numbers. Let, for example, they be $\underline{d}_1\bar{d}_2$ and $\bar{d}_1\bar{d}_2$. We have thus found that

$$\operatorname{Arg}\min\left\{x_1 \cdot x_2 \mid x_1 \in \left[\underline{d}_1;\ \bar{d}_1\right],\ x_2 \in \left[\underline{d}_2;\ \bar{d}_2\right]\right\} = \begin{pmatrix} \underline{d}_1 \\ \bar{d}_2 \end{pmatrix},$$

$$\operatorname{Arg}\max\left\{x_1 \cdot x_2 \mid x_1 \in \left[\underline{d}_1;\ \bar{d}_1\right],\ x_2 \in \left[\underline{d}_2;\ \bar{d}_2\right]\right\} = \begin{pmatrix} \bar{d}_1 \\ \bar{d}_2 \end{pmatrix}.$$

If the minimal values over the set $\{\bar{d}_1\bar{d}_2,\ \underline{d}_1\bar{d}_2,\ \bar{d}_1\underline{d}_2,\ \underline{d}_1\underline{d}_2\}$ were two products simultaneously, say, $\underline{d}_1\underline{d}_2$ and $\underline{d}_1\bar{d}_2$, then we would conclude the following:

$$\operatorname{Arg}\min\left\{x_1 \cdot x_2 \mid x_1 \in \left[\underline{d}_1;\ \bar{d}_1\right],\ x_2 \in \left[\underline{d}_2;\ \bar{d}_2\right]\right\} = \begin{pmatrix} \underline{d}_1 \\ \left[\underline{d}_2;\ \bar{d}_2\right] \end{pmatrix},$$

that is, the set $\operatorname{Arg}\min x_1 \cdot x_2$ consists of the whole face of the rectangle $[\underline{d}_1;\ \bar{d}_1] \times [\underline{d}_2;\ \bar{d}_2]$ of "input data" in this case.

Further, the value of any rational expression on an interval vector may be computed by a finite number of interval additions, subtractions, multiplications and divisions. If each variable appears only once in $F(x)$ and only to the first power, then, recurrently tracing evolution of the intervals' endpoints, we can reveal that collection of endpoints of the initial intervals $[\underline{d}_1;\ \overline{d}_1], [\underline{d}_2;\ \overline{d}_2], \ldots, [\underline{d}_n;\ \overline{d}_n]$, in which $\underline{F(\mathbf{D})}$ and $\overline{F(\mathbf{D})}$ are attained.

It is fairly simple to realize that all stated above holds true when the expression $F(x)$ that has single occurrence of each variable is constructed not only of the four arithmetic operations, but contains occurrences $\Lambda_1, \Lambda_2, \ldots$ of any other functions as well. We only need, when taking the "natural interval extension" of F, to change $\Lambda_1, \Lambda_2, \ldots$ for occurrences of the respective optimal interval extensions. For example, if, in the expressions

$$\Psi(x) = R - \left| M - \sum_{j=1}^{n} x_j t_j \right| \qquad \text{and} \qquad \Theta(x) = \sum_{j=1}^{n} |x_j|,$$

one changes all arithmetic operations by their interval counterparts and instead of the absolute value function its optimal interval extension

$$\operatorname{abs}\left(\left[\underline{d};\ \overline{d}\right]\right) = \left[\langle\left[\underline{d};\ \overline{d}\right]\rangle; \left|\left[\underline{d};\ \overline{d}\right]\right|\right]$$

is taken, then the substitution of the variables $x_1, x_2, \ldots, x_n$ for the intervals of their possible variations $[\underline{d}_1;\ \overline{d}_1], [\underline{d}_2;\ \overline{d}_2], \ldots, [\underline{d}_n;\ \overline{d}_n]$ gives the exact ranges of values for $\Psi(x)$ and $\Theta(x)$ over any rectangle $\mathbf{D} = [\underline{d}_1;\ \overline{d}_1] \times [\underline{d}_2;\ \overline{d}_2] \times \cdots \times [\underline{d}_n;\ \overline{d}_n] \in I\mathbb{R}^n$.

Below, we shall be interested mainly in the sets Arg $\underline{\Psi(\mathbf{D})}$ and Arg $\overline{\Theta(\mathbf{D})}$ for an interval vector $\mathbf{D} \in I\mathbb{R}^n$. The problem of their computation is by no means harder than that for purely rational expressions, since we know how to find Arg min and Arg max of the absolute value function which occur in $\Psi(x)$ and $\Theta(x)$ aside from the basic arithmetical operations. It can be solved by that same "tracing endpoints" technique. For example, the algorithm for computing $\overline{\Theta(\mathbf{D})}$ and Arg $\overline{\Theta(\mathbf{D})}$ written in informal ALGOL looks as follows.

max := 0;
for $j := 1$ **step** 1 **until** n **do**
  if $|\underline{d}_j| \leqslant |\overline{d}_j|$ **then begin** $max := max + \overline{d}_j$; $(\text{Arg}\ \overline{\Theta(\mathbf{D})})_j := \overline{d}_j$ **end**
  else begin $max := max + \underline{d}_j$; $(\text{Arg}\ \overline{\Theta(\mathbf{D})})_j := \underline{d}_j$ **end**
$\overline{\Theta(\mathbf{D})} := max$;

Of course, this algorithm is simplified for the sake of obviousness and does not take into account the fact that in reality the set Arg $\overline{\Theta(\mathbf{D})}$ may have quite nontrivial structure. In particular, it may be disconnected if $\underline{d}_j = -\overline{d}_j \neq 0$ for some $j \in \{1, 2, \ldots, n\}$. The set Arg $\underline{\Psi(\mathbf{D})}$ is also disconnected if

$$\overline{M - \sum_{j=1}^{n} \left[\underline{d}_j;\ \overline{d}_j\right] t_j} = \overline{\sum_{j=1}^{n} \left[\underline{d}_j;\ \overline{d}_j\right] t_j - M}.$$

In any case, separate connected components of the sets Arg $\overline{\Theta(\mathbf{D})}$ and Arg $\underline{\Psi(\mathbf{D})}$ are represented as direct products $\mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_n$, where $\mathcal{G}_j$ is either a vertex of the rectangle $\mathbf{D}$ or its entire edge.

Let us agree to take from now on as Arg $\underline{\Psi(\mathbf{D})}$ (or Arg $\overline{\Theta(\mathbf{D})}$) a connected component of the set of points furnishing with $\underline{\Psi(\mathbf{D})}$ ($\overline{\Theta(\mathbf{D})}$ respectively) and it does not matter which one exactly. Also, it is worth noting that for sets $\mathscr{G}$, $\mathscr{H} \subseteq \mathbb{R}^n$ with the direct product structure, that is, when $\mathscr{G} = \mathscr{G}_1 \times \mathscr{G}_2 \times \cdots \times \mathscr{G}_n$, $\mathscr{G}_j \subseteq \mathbb{R}$, and $\mathscr{H} = \mathscr{H}_1 \times \mathscr{H}_2 \times \ldots \times \mathscr{H}_n$, $\mathscr{H}_j \subseteq \mathbb{R}$, the condition $(\mathscr{G} \cap \mathscr{H} = \emptyset)$ is equivalent to $(\mathscr{G}_j \cap \mathscr{H}_j = \emptyset)$ for at least one of $j \in \{1, 2, \ldots, n\}$. We shall substantially avail ourselves of this property later.

After such preliminary preparations, we turn to the actual computation of

$$\min\{\Phi(x) \mid x \in \text{vert } \mathbf{D}\} = \min\left\{ \frac{\Psi(x)}{\Theta(x)} \,\middle|\, x \in \text{vert } \mathbf{D} \right\}$$

$$= \min_{\substack{x_j \in \{\underline{d}_j,\, \overline{d}_j\},\\ j = 1, 2, \ldots, n}} \left\{ \frac{R - \left| M - \sum_{j=1}^n x_j t_j \right|}{\sum_{j=1}^n |x_j|} \right\}, \qquad (28)$$

where R, M, $t_1, t_2, \ldots, t_n$ are some known constants. As we have already noted, the simplest way to estimate (28) from below is to construct the natural interval extension $\Phi(\mathbf{D})$ for the minimized function $\Phi(x)$ over *entire* $\mathbf{D}$. Its left endpoint $\underline{\Phi(\mathbf{D})}$ that coincides with

$$\frac{\underline{\Psi(\mathbf{D})}}{\overline{\Theta(\mathbf{D})}} = \frac{\min\{\Psi(x) \mid x \in \mathbf{D}\}}{\max\{\Theta(x) \mid x \in \mathbf{D}\}}$$

gives the required lower estimate for $\min\{\Phi(x) \mid x \in \text{vert } \mathbf{D}\}$.

We find the sets Arg $\underline{\Psi(\mathbf{D})}$ and Arg $\overline{\Theta(\mathbf{D})}$. One may come across the following mutually exclusive situations:

(i) $\text{Arg } \underline{\Psi(\mathbf{D})} \cap \text{Arg } \overline{\Theta(\mathbf{D})} \neq \emptyset$ or

(ii) $\text{Arg } \underline{\Psi(\mathbf{D})} \cap \text{Arg } \overline{\Theta(\mathbf{D})} = \emptyset$.

In the first case, any point of the intersection $\text{Arg } \underline{\Psi(\mathbf{D})} \cap \text{Arg } \overline{\Theta(\mathbf{D})}$ must contain a point of vert $\mathbf{D}$ that provides the fraction $\Psi(x)/\Theta(x)$ with the global minimum over $\mathbf{D}$, and so the minimization problem for (28) is successfully solved. If $\text{Arg } \underline{\Psi(\mathbf{D})} \cap \text{Arg } \overline{\Theta(\mathbf{D})} = \emptyset$, then

$$\underline{\Phi(\mathbf{D})} = \frac{\min\{\Psi(x) \mid x \in \mathbf{D}\}}{\max\{\Theta(x) \mid x \in \mathbf{D}\}} \geqslant \min\left\{ \frac{\Psi(x)}{\Theta(x)} \,\middle|\, x \in \mathbf{D} \right\}$$

and there exists an index $k \in \{1, 2, \ldots, n\}$ such that the k-th components of the sets Arg $\underline{\Psi(\mathbf{D})}$ and Arg $\overline{\Theta(\mathbf{D})}$ have no common values:

$$(\text{Arg } \underline{\Psi(\mathbf{D})})_k \cap (\text{Arg } \overline{\Theta(\mathbf{D})})_k = \emptyset.$$

Henceforth we shall refer to such components of the rectangle $\mathbf{D}$ as *incidental*.

If k is the index of an incidental component (so, in particular, $\underline{d}_k \neq \overline{d}_k$), put

$$\mathbf{D}' := [\underline{d}_1; \overline{d}_1] \times \cdots \times \underline{d}_k \times \cdots \times [\underline{d}_n; \overline{d}_n],$$

$$\mathbf{D}'' := [\underline{d}_1; \overline{d}_1] \times \cdots \times \overline{d}_k \times \cdots \times [\underline{d}_n; \overline{d}_n].$$

We will speak that the rectangles $\mathbf{D}'$ and $\mathbf{D}''$ are *descendants* of $\mathbf{D}$ and call the very procedure of their generating the *subdivision* of the initial rectangle $\mathbf{D}$. Inclusion monotonicity of the interval arithmetic [1,2,3,9,11,17] implies

$$\underline{\Phi(\mathbf{D})} \leqslant \underline{\Phi(\mathbf{D}')} \quad \text{and} \quad \underline{\Phi(\mathbf{D})} \leqslant \underline{\Phi(\mathbf{D}'')},$$

and therefore

$$\underline{\Phi(\mathbf{D})} \leqslant \min\{\underline{\Phi(\mathbf{D}')}, \underline{\Phi(\mathbf{D}'')}\}.$$

In fact, this inequality is strict under our assumption, since the sets Arg $\underline{\Psi(\mathbf{D})}$ and Arg $\overline{\Theta(\mathbf{D})}$ get into different rectangles-descendants as the result of subdivision of $\mathbf{D}$ (if they may be represented as direct products). For example, if we set for the sake of definiteness that

$$\text{Arg}\ \underline{\Psi(\mathbf{D})} \subseteq \mathbf{D}' \quad \text{and} \quad \text{Arg}\ \overline{\Theta(\mathbf{D})} \subseteq \mathbf{D}''$$

then

$$\underline{\Psi(\mathbf{D})} = \underline{\Psi(\mathbf{D}')} \quad \text{and} \quad \overline{\Theta(\mathbf{D})} = \overline{\Theta(\mathbf{D}'')},$$

while

$$\underline{\Psi(\mathbf{D}'')} > \underline{\Psi(\mathbf{D})} \quad \text{and} \quad \overline{\Theta(\mathbf{D}')} < \overline{\Theta(\mathbf{D})}.$$

Consequently,

$$\frac{\underline{\Psi(\mathbf{D})}}{\overline{\Theta(\mathbf{D})}} < \frac{\underline{\Psi(\mathbf{D}')}}{\overline{\Theta(\mathbf{D}')}} \quad \text{and} \quad \frac{\underline{\Psi(\mathbf{D})}}{\overline{\Theta(\mathbf{D})}} < \frac{\underline{\Psi(\mathbf{D}'')}}{\overline{\Theta(\mathbf{D}'')}},$$

that is, in reality

$$\underline{\Phi(\mathbf{D})} < \min\{\underline{\Phi(\mathbf{D}')}, \underline{\Phi(\mathbf{D}'')}\}.$$

Let

$$\min\{\Phi(x) \mid x \in \text{vert}\ \mathbf{D}\} = \Phi(\Pi)$$

for some vertex $\Pi \in \text{vert}\ \mathbf{D}$, $\Pi \in \mathbb{R}^n$. If Π' and Π'' are points obtained from Π by substituting its k-th component for $\underline{d}_k$ and $\overline{d}_k$ respectively, that is, for the left and right endpoints of the interval $\mathbf{d}_k$, then again making use of the inclusion monotonicity of the interval arithmetic, we find out

$$\underline{\Phi(\mathbf{D}')} \leqslant \Phi(\Pi') \quad \text{and} \quad \underline{\Phi(\mathbf{D}'')} \leqslant \Phi(\Pi''),$$

so that

$$\min\{\underline{\Phi(\mathbf{D}')}, \underline{\Phi(\mathbf{D}'')}\} \leqslant \min\{\Phi(\Pi'), \Phi(\Pi'')\} = \Phi(\Pi).$$

Because of this,

$$\underline{\Phi(\mathbf{D})} < \min\{\underline{\Phi(\mathbf{D}')}, \underline{\Phi(\mathbf{D}')}\} \leqslant \min\left\{ \frac{\Psi(x)}{\Theta(x)} \,\middle|\, x \in \text{vert}\ \mathbf{D} \right\}.$$

The presented reasoning is thus a practical prescription for sharpening the lower estimate of (28). The subdivision procedure may be repeated with the rectangles-descendants $\mathbf{D}'$ and $\mathbf{D}''$ to get refined approximation of $\min\{\Phi(x) \mid x \in \mathbf{D}\}$ as the minimum of all obtained estimates, then to subdivide again the descendants of $\mathbf{D}'$ and $\mathbf{D}''$ and to further sharpen the estimate for (28) and so on. It advisable to arrange this process of successive refinement utilizing the general scheme of the "branch-and-bound" method (see, e.g., [15]): bisection of a rectangle $\mathbf{D}$ is nothing but a decomposition of the problem into subproblems, that is, producing "branches", while calculations of $\underline{\Phi(\mathbf{P})}$, $\mathbf{P} \subseteq \mathbf{D}$, are simply estimations of the objective function's "bounds" over these "branches".

We shall keep the set of all rectangles $\mathbf{P}$ resulted in subdividing (that is, actually, of all active subproblems of the original problem) together with all their related estimates in the form of an *ordered list* L consisting of sextuples

$$\left(\mathbf{P},\ \underline{\Phi(\mathbf{P})},\ \underline{\Psi(\mathbf{P})},\ \overline{\Theta(\mathbf{P})},\ \operatorname{Arg}\underline{\Psi(\mathbf{P})},\ \operatorname{Arg}\overline{\Theta(\mathbf{P})}\right),$$

$\mathbf{P} \in \mathbb{IR}^n$, $\mathbf{P} \subseteq \mathbf{D}$. As usual, the first record of the list L, that has the smallest estimate $\underline{\Phi(\mathbf{P})}$ to the beginning of the current step of the algorithm, is referred to as the *leading record*. At the start of the successive refinement algorithm,

$$L = \left\{\left(\mathbf{D},\ \underline{\Phi(\mathbf{D})},\ \underline{\Psi(\mathbf{D})},\ \overline{\Theta(\mathbf{D})},\ \operatorname{Arg}\underline{\Psi(\mathbf{D})},\ \operatorname{Arg}\overline{\Theta(\mathbf{D})}\right)\right\}$$

and then the list is modified at each step according to the following instructions:

Algorithm 5

1. If

   $$\operatorname{Arg}\underline{\Psi(\mathbf{Q})} \cap \operatorname{Arg}\overline{\Theta(\mathbf{Q})} \neq \emptyset$$

   in the leading rectangle $\mathbf{Q}$, then stop computation.
2. Choose an incidental component $\mathbf{Q}_\iota$ in the leading rectangle. Bisect $\mathbf{Q}$ to descendants $\mathbf{Q}'$ and $\mathbf{Q}''$ so that $\mathbf{Q}'_\iota$, and $\mathbf{Q}''_\iota$, are the opposite endpoints of the interval $\mathbf{Q}_\iota$, and

   $$\operatorname{Arg}\underline{\Psi(\mathbf{Q})} \subseteq \mathbf{Q}' \text{ and } \operatorname{Arg}\overline{\Theta(\mathbf{Q})} \subseteq \mathbf{Q}''.$$

3. Delete in the list L the late leading record

   $$\left(\mathbf{P},\ \underline{\Phi(\mathbf{P})},\ \underline{\Psi(\mathbf{P})},\ \overline{\Theta(\mathbf{P})},\ \operatorname{Arg}\underline{\Psi(\mathbf{P})},\ \operatorname{Arg}\overline{\Theta(\mathbf{P})}\right).$$

4. Calculate $\overline{\Theta(\mathbf{Q}')}$ and $\operatorname{Arg}\overline{\Theta(\mathbf{Q}')}$, $\underline{\Psi(\mathbf{Q}'')}$ and $\operatorname{Arg}\underline{\Psi(\mathbf{Q}'')}$.
5. Put $\underline{\Phi(\mathbf{Q}')} := \underline{\Psi(\mathbf{Q})}/\overline{\Theta(\mathbf{Q}')}$ and $\underline{\Phi(\mathbf{Q}'')} := \underline{\Psi(\mathbf{Q}'')}/\overline{\Theta(\mathbf{Q})}$.
6. Enter the records $\left(\mathbf{Q}',\ \underline{\Phi(\mathbf{Q}')},\ \underline{\Psi(\mathbf{Q}')},\ \overline{\Theta(\mathbf{Q}')},\ \mathit{Arg}\,\underline{\Psi(\mathbf{Q}')},\ \mathit{Arg}\,\overline{\Theta(\mathbf{Q}')}\right)$ and $\left(\mathbf{Q}'',\ \underline{\Phi(\mathbf{Q}'')},\ \underline{\Psi(\mathbf{Q}'')},\ \overline{\Theta(\mathbf{Q}'')},\ \mathit{Arg}\,\underline{\Psi(\mathbf{Q}'')},\ \mathit{Arg}\,\overline{\Theta(\mathbf{Q}'')}\right)$ into the list L in the proper order (of increasing the second member).

As the result of this algorithm's fulfillment, a nondecreasing sequence of estimates $\underline{\Phi(\mathbf{Q})}$ for the leading rectangles $\mathbf{Q}$ is obtained, that better and better approximates the sought-for $\min\{\Phi(x) \mid x \in \operatorname{vert}\mathbf{D}\}$ from below. It is finite: if T components ($T \leqslant n$) have nonzero width in the initial rectangle $\mathbf{D}$, then, as is easily seen, the leading rectangle will become a singleton Π after at most 2^T steps of Algorithm 5 and $\Phi(\Pi) = \min\{\Phi(x) \mid x \in \operatorname{vert}\mathbf{D}\}$.

The idea of such algorithms for global optimization is undeniably not new, the similar methods were studied in [3,9,17] and many other works. However, when designing Algorithm 5 we tried to make the best use of the problem's structure, namely of the fact that the sought-for $\min\{\Phi(x) \mid x \in \mathbf{D}\}$ is reached in a vertex of the rectangle $\mathbf{D}$. In particular, we subdivide the leading rectangle to descendants which are not even its halves, but mere two its opposite faces with the smaller dimension. The other original feature of Algorithm 5 is that the subdivision (bisection) of the leading rectangles is executed not in all components, but only in incidental, that is, in such ones whose shortening solely refine the estimates of the objective function. In other words, we do not slip to disordered partitioning of the leading rectangles, but subdivide them so as to ensure a guaranteed sharpening of the estimate for (28). Finally, Algorithm 5 more completely utilizes the information about its preceding work, at the price of some extension of the records kept in the list L. Thanks to the last expedient, we calculate the objective function's estimate really only once for two rectangles-descendants.

Let us proceed to elaborate a more practical computational procedure on the basis of Algorithm 5. In principle, we may improve the simplest algorithm along the following standard list of modifications (see, e.g., [3,17] and others):

- tracing values of the objective function at some points of boxes along with evaluating over entire boxes enables one to control the precision of the current approximation to the sought-for optimum and to delete useless pairs (that never become leading) from the list L,
- after revealing monotonicity of the objective function in some variables, one reduces the dimension of boxes from the list L,
- based upon local characteristics of the objective function, one employs minimization procedures in appropriate boxes which are more efficient than bisection,
- one construct a higher quality (more accurate) inclusion function for the objective function and so on.

We shall apply only the first ("midpoint test") and the second ("monotonicity test") of all the presented advances the more so, that they turn out to be aptly mutually complementary. Besides, to make improvement of the estimate of the sought for minimum more weighted, it is advisable to make use of the well-known heuristic recommendation: the leading rectangle is bisected only upon the longest of the incidental components at each step of the algorithm.

The classical "midpoint test" scheme as applied to Algorithm V is implemented as follows. Each time one calculates $\Phi(\text{mid}\,\mathbf{P})$ a long with $\underline{\Phi(\mathbf{P})}$ and a real parameter ϕ is connected with the algorithm which is equal to the smallest one among the values $\Phi(\text{mid}\,\mathbf{P})$ for all rectangles $\mathbf{P}$ ever been generated by the algorithm up to the current step. Then

$$\min\{\Phi(x) \mid x \in \mathbf{D}\} \leqslant \phi,$$

and all records $\left(\mathbf{P},\, \underline{\Phi(\mathbf{P})},\, \underline{\Psi(\mathbf{P})},\, \overline{\Theta(\mathbf{P})},\, \mathit{Arg}\, \underline{\Psi(\mathbf{P})},\, \mathit{Arg}\, \overline{\Theta(\mathbf{P})}\right)$ that satisfy

$$\phi < \underline{\Phi(\mathbf{P})} \tag{29}$$

may be removed from the list L without any effect on the algorithm's performance. However, substantial difficulties obstruct the realization of this idea. The thing is that the leading rectangle's diameters does not necessarily tend to zero in Algorithm 5, since the rectangles are partitioned only in incidental components. Such a peculiarity of Algorithm 5 is undoubtedly positive inasmuch as it is this property that ensures economy and purposefulness of computation. On the other hand, it causes that the difference $(\Phi(\text{mid}\,\mathbf{P}) - \underline{\Phi(\mathbf{P})})$ may remain greater than some positive number even for the leading rectangles $\mathbf{Q}$. The numerical experiments show that then, as a rule, the inequality (29) never holds and we may recognize no one of the records of L as useless. Thus, there is a necessity to supplement Algorithm 5, apart from the "midpoint test", with a procedure that reduces the component's size whenever it is incidental or not. In the point at issue, this can be a "contraction" of the rectangle on the components, in which the monotonicity of the objective function is revealed.

The objective function $\Phi(x)$ is nonsmooth, but continuous and almost everywhere differentiable. So we examine its monotonicity in separate variables over rectangles $\mathbf{P} \subseteq \mathbb{R}^n$ by the standard way, that is, through evaluating the sign of interval extensions of the derivatives $\partial\Phi(x)/\partial x_j$ over $\mathbf{P}$. Since $\Phi(x) = \Psi(x)/\Theta(x)$, we have

$$\Phi'(x) = \frac{\Psi'(x)\Theta(x) - \Psi(x)\Theta'(x)}{(\Theta(x))^2}.$$

Furthermore,

$$\begin{aligned} \frac{\partial\Psi(x)}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(R - \left|M - \sum_{j=1}^{n} x_j t_j\right|\right) \\ &= -\operatorname{sgn}\left(M - \sum_{j=1}^{n} x_j t_j\right) \cdot \frac{\partial}{\partial x_j}\left(M - \sum_{j=1}^{n} x_j t_j\right) \\ &= t_j \cdot \operatorname{sgn}\left(M - \sum_{j=1}^{n} x_j t_j\right), \end{aligned}$$

$$\frac{\partial\Theta(x)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\sum_{j=1}^{n} |x_j|\right) = \operatorname{sgn} x_j,$$

and so

$$\begin{aligned} &\Psi'(x)\Theta(x) - \Psi(x)\Theta'(x) \\ &\quad = t_j \cdot \operatorname{sgn}\left(M - \sum_{j=1}^{n} x_j t_j\right) \cdot \left(\sum_{j=1}^{n} |x_j|\right) - \left(M - \sum_{j=1}^{n} x_j t_j\right) \cdot \operatorname{sgn} x_j. \end{aligned} \tag{30}$$

Since the optimal interval extension of the sign function is

$$\operatorname{sgn}([\underline{d}; \bar{d}]) = \begin{cases} 1, & \text{if } 0 < \underline{d}, \\ [0; 1], & \text{if } \underline{d} = 0 < \bar{d}, \\ [-1; 1], & \text{if } \underline{d} < 0 < \bar{d}, \\ [-1; 0], & \text{if } \underline{d} < 0 = \bar{d}, \\ -1, & \text{if } \bar{d} < 0, \end{cases}$$

one can easily find the natural interval extension for the expression (30), whose sign coincides with that of the natural interval extension of the derivatives $\partial\Phi(x)/\partial x_j$, $j = 1, 2, \dots, n$.

Finally, for practical calculations of the size of interval solutions to the linear tolerance problem, we recommend the following

Algorithm 6. Put $\phi := \Phi(\text{mid } \mathbf{D})$ and

$$L = \left\{\left(\mathbf{D},\ \underline{\Phi(\mathbf{D})},\ \underline{\Psi(\mathbf{D})},\ \overline{\Theta(\mathbf{D})},\ \text{Arg}\ \underline{\Psi(\mathbf{D})},\ \text{Arg}\ \overline{\Theta(\mathbf{D})}\right)\right\}$$

1. If in the leading rectangle $\mathbf{Q}$

   $$\text{Arg}\ \underline{\Psi(\mathbf{Q})} \cap \text{Arg}\ \overline{\Theta(\mathbf{Q})} \neq \emptyset$$

   or if $(\phi - \underline{\Phi(\mathbf{Q})}) \leqslant \epsilon$, then stop computation.
2. Choose the incidental component $\mathbf{Q}_\iota$ of the largest length in the leading rectangle and subdivide $\mathbf{Q}$ to the descendants $\mathbf{Q}'$ and $\mathbf{Q}''$ so that $\mathbf{Q}'_\iota$ and $\mathbf{Q}''_\iota$ are the opposite endpoints of the interval $\mathbf{Q}_\iota$ and

   $$\text{Arg}\ \underline{\Psi(\mathbf{Q})} \subseteq \mathbf{Q}' \quad \text{and} \quad \text{Arg}\ \overline{\Theta(\mathbf{Q})} \subseteq \mathbf{Q}''$$

3. Calculate the natural interval extension of the expression (30) over $\mathbf{Q}'$ for all $j \in \{1, 2, \ldots, n\}$ such that rad $\mathbf{Q}'_j \neq 0$. If it is a nonnegative interval, then substitute $\mathbf{Q}'_j$ for its left endpoint $\underline{\mathbf{Q}'}_j$, and if it is a nonpositive one, then for its right endpoint $\overline{\mathbf{Q}'}_j$.
   We keep the former designation $\mathbf{Q}'$ for the rectangle resulted from this procedure.
4. Do with the rectangle $\mathbf{Q}''$ that same as in the preceding item.
5. Calculate $\overline{\Theta(\mathbf{Q}')}$ and Arg $\overline{\Theta(\mathbf{Q}')}$, $\underline{\Psi(\mathbf{Q}'')}$ and Arg $\underline{\Psi(\mathbf{Q}'')}$.
6. Put $\underline{\Phi(\mathbf{Q}')} := \underline{\Psi(\mathbf{Q})}/\overline{\Theta(\mathbf{Q}')}$ and $\underline{\Phi(\mathbf{Q}'')} := \underline{\Psi(\mathbf{Q}'')}/\overline{\Theta(\mathbf{Q})}$.
7. If $\underline{\Phi(\mathbf{Q}')} \leqslant \phi$, then enter the record $\left(\mathbf{Q}',\ \underline{\Phi(\mathbf{Q}')},\ \underline{\Psi(\mathbf{Q})},\ \overline{\Theta(\mathbf{Q}')},\ \mathit{Arg}\ \underline{\Psi(\mathbf{Q})},\ \mathit{Arg}\ \overline{\Theta(\mathbf{Q}')}\right)$ into the list L in the proper order (of increasing the second member).
8. If $\underline{\Phi(\mathbf{Q}'')} \leqslant \phi$, then enter the record $(\mathbf{Q}'',\ \underline{\Phi(\mathbf{Q}'')},\ \underline{\Psi(\mathbf{Q}'')},\ \overline{\Theta(\mathbf{Q})},\ \text{Arg}\ \underline{\Psi(\mathbf{Q}'')},\ \text{Arg}\ \overline{\Theta(\mathbf{Q})})$ into the list L in the proper order.
9. Delete the late leading record $\left(\mathbf{Q},\ \underline{\Phi(\mathbf{Q})},\ \underline{\Psi(\mathbf{Q})},\ \overline{\Theta(\mathbf{Q})},\ \mathit{Arg}\ \underline{\Psi(\mathbf{Q})},\ \mathit{Arg}\ \overline{\Theta(\mathbf{Q})}\right)$ from the list L.
10. Calculate $\omega := \min\{\Phi(\text{mid } \mathbf{Q}'), \Phi(\text{mid } \mathbf{Q}'')\}$.
11. If $\phi > \omega$, then put $\phi := \omega$ and clean the list L, i.e., delete from it all records $\left(\mathbf{P},\ \underline{\Phi(\mathbf{P})},\ \underline{\Psi(\mathbf{P})},\ \overline{\Theta(\mathbf{P})},\ \mathit{Arg}\ \underline{\Psi(\mathbf{P})},\ \mathit{Arg}\ \overline{\Theta(\mathbf{P})}\right)$ such that $\underline{\Phi(\mathbf{P})} > \phi$.

In the second instruction, ε is a prescribed absolute accuracy of the result. For large dimensions, the practical work with this algorithm shows, however, that more often its stop is due to exhaustion of the computer resources (primarily, of the time). In such cases, we still obtain the answer to the problem: it will be the last computed leading estimate $\underline{\Phi(\mathbf{Q})}$.

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