Interval Kalman Filtering

GUANRONG CHEN, Senior Member, IEEE
JIANRONG WANG
LEANG S. SHIEH, Senior Member, IEEE
University of Houston

The classical Kalman filtering technique is extended to interval linear systems with the same statistical assumptions on noise, for which the classical technique is no longer applicable. Necessary interval analysis, particularly the notion of interval expectation, is reviewed and introduced. The interval Kalman filter (IKF) is then derived, which has the same structure as the classical algorithm, using no additional analysis or computation from such as H^∞-mathematics. A suboptimal IKF is suggested next, for the purpose of real-time implementation. Finally, computer simulations are shown to compare the new interval Kalman filtering algorithm with the classical Kalman filtering scheme and some other existing robust Kalman filtering methods.

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Authors' address: Dept. of Electrical and Computer Engineering, University of Houston, Houston, TX 77204-4793.

I. INTRODUCTION

Robust estimation, or robust filtering, for uncertain linear systems has been investigated under different conditions in the last two decades (see, for instance, [8, 9] and the references therein). In particular, robust Kalman filtering with respect to uncertain linear systems is still an active research topic that attracts increasing interest, on which several approaches have been proposed: using H_∞-criteria [4, 12, 14, 16, 17], set-valued estimations [5, 6, 11], and interval systems analysis [7, 13, 15, 18].

It is well known that optimal estimates are given by a conditional expectation of the unknown random variable (or random vector), under the condition that the data to be used were given [2]. The standard Kalman filtering scheme was derived directly from this statistical criterion, and is hence optimal in the sense that it exactly (not approximately) satisfies the criterion, and so provides a precise linear, unbiased and minimum-error-variance estimate at each recursive step throughout the filtering process. However, to the best of our knowledge, the aforementioned approaches to robust filtering essentially suggest approximate estimations, such as the best solution in the worst case, and do not provide theoretically optimal estimations (in the statistical sense of conditional expectation) for each linear system existing within the uncertain bounds. Besides, these approaches have more or less lost the fundamental characteristic of the standard Kalman filter (SKF) that satisfies the familiar statistical conditions and criteria.

In this work we develop an exact and optimal Kalman filtering algorithm based on interval conditional expectation for interval linear systems, for which the classical algorithm is no longer applicable. The new interval Kalman filtering scheme has the same structure as the classical algorithm, using no additional analysis or computation from such as H^∞-mathematics. The interval Kalman filtering algorithm thus preserves both the statistical optimality and the recursive computational scheme of the SKF.

In Section II, we first review and introduce some fundamental interval analysis and the notion of interval expectation in statistics, which are necessary for the development of the new algorithm in the sequel. Then, the interval Kalman filter (IKF) is derived in Section III, with brief but sufficient mathematical justification. A suboptimal IKF is suggested in Section IV, for the purpose of real-time implementation and applications. Finally, computer simulations are shown in Section V, to compare the new algorithm with the classical Kalman filtering scheme and some other existing robust Kalman filtering methods.

II. PRELIMINARIES

In this section, we first review and develop some preliminary results on interval arithmetic and interval analysis that are needed throughout the paper. Most of these preliminary results can be found in [1, 10], perhaps in different formats.

A. Interval Mathematics

Let S_1 and S_2 be real intervals in $\Re = (-\infty, \infty)$ and $f : S_1 \to S_2$ be an arbitrarily, ordinary one-variable real-valued (i.e., point-to-point) function. Denote by $\sum_{S_1}$ and $\sum_{S_2}$ the families of all subintervals of S_1 and S_2, respectively. The interval-to-interval function $f_I : \sum_{S_1} \to \sum_{S_2}$ defined by

$$f_I(X) = \left\{ f(x) \in S_2 : x \in X,\ X \in \sum_{S_1} \right\}$$

is called the *united extension* of the point-to-point function f on S_1. Obviously, the range is

$$f_I(X) = \bigcup_{x \in X} \{f(x)\}$$

which is the union of all the subintervals of S_2 that contain a singleton $f(x)$ for some $x \in X$.

It follows immediately from definition that the united extension $f_I : \sum_{S_1} \to \sum_{S_2}$ has the following property:

$$X, Y \in \sum_{S_1} \quad \text{and} \quad X \subseteq Y \quad \Rightarrow \quad F_I(X) \subseteq f_I(Y).$$

In general, an interval-to-interval function F of n-variables $X_1, \ldots, X_n$ is said to have the *inclusion monotonic property* if

$$X_i \subseteq Y_i \qquad \forall \quad i = 1, \ldots, n$$
$$\Rightarrow F(X_1, \ldots, X_n) \subseteq F(Y_1, \ldots, Y_n).$$

Clearly, all the united extensions have the inclusion monotonic property.

Since interval arithmetic functions are united extensions of the real arithmetic functions: addition, subtraction, multiplication, and division $(+, -, \cdot, /)$, interval arithmetic is inclusion monotonic:

$$X_1 \subseteq Y_1 \qquad \text{and} \qquad X_2 \subseteq Y_2$$

together imply

a) $X_1 + X_2 \subseteq Y_1 + Y_2$

b) $X_1 - X_2 \subseteq Y_1 - Y_2$

c) $X_1 \cdot X_2 \subseteq Y_1 \cdot Y_2$

d) $X_1 / X_2 \subseteq Y_1 / Y_2$,

provided that the operations are well defined.

An interval-to-interval function is called an *interval function* for simplicity. *Interval vectors* and *interval matrices* are similarly defined. An interval function is said to be *rational*, and so is called a *rational interval function*, if its values are defined by a finite sequence of interval arithmetic operations. Examples of rational interval functions include $X + Y^2 + Z^3$ and $(X^2 + Y^2)/Z$, etc., for intervals X, Y, and Z, where in the latter $0 \notin Z$.

It follows from the transitivity of the partially ordered relation $\subseteq$ that all the rational interval functions are inclusion monotonic. This can be verified by a finite mathematical induction. Note, however, that not all the interval functions have the inclusion monotonic property.

Next, let $f = f(x_1, \ldots, x_n)$ be an ordinary n-variable real-valued function, and let $X_1, \ldots, X_n$ be real intervals. An interval function $F = F(X_1, \ldots, X_n)$ is said to be an *interval extension* of f if

$$F(x_1, \ldots, x_n) = f(x_1, \ldots, x_n),$$
$$\forall \quad x_i \in X_i, \qquad i = 1, \ldots, n.$$

Note also that not all the interval extensions have the inclusion monotonic property.

The following fundamental result is well known [1].

THEOREM A *If F is an inclusion monotonic interval extension of f, then the united extension f_I of f satisfies*

$$f_I(X_1, \ldots, X_n) \subseteq F(X_1, \ldots, X_n).$$

Since rational interval functions are inclusion monotonic, we have the following result.

COROLLARY B *If F is a rational interval function and is an interval extension of f, then*

$$f_I(X_1, \ldots, X_n) \subseteq F(X_1, \ldots, X_n).$$

This corollary can provide us with a means of finite evaluation of upper and lower bounds on the value-range of an ordinary rational function over an n-dimensional rectangle in $\Re^n$.

B. Interval Expectation

Now, let $f(x)$ be an ordinary function defined on a real interval X, such that it satisfies the ordinary Lipschitz condition

$$|f(x) - f(y)| \le L|x - y|$$

for some constant L, independent of $x, y \in X$. Then the united extension f_I of f is said to be a *Lipschitz interval extension* of f over X.

Let $C(X)$ be a class of functions defined on X that are most commonly used in computation, such as the four arithmetic functions $(+, -, \cdot, /)$ and the elementary type of functions like $\exp\{\cdot\}$, $\ln(\cdot)$, $\sqrt{\cdot}$, etc. We only use some of such commonly used functions throughout this work. Let N be a positive integer and

subdivide an interval $[a,b] \subseteq X$ into N subintervals $X_1 = [\mathbf{X}_1, \overline{X}_1], \ldots, X_N = [\mathbf{X}_N, \overline{X}_N]$, such that

$$a = \mathbf{X}_1 < \overline{X}_1 = \mathbf{X}_2 < \overline{X}_2 = \cdots = \mathbf{X}_N < \overline{X}_N = b.$$

Moreover, for any $f \in C(X)$, let F be an inclusion monotonic and Lipschitz interval extension of f defined on all X_i, $i = 1, \ldots, N$. Using the notation

$$S_N(F;[a,b]) = \frac{b-a}{N} \sum_{i=1}^{N} F(X_i)$$

we have

$$\int_a^b f(t)\,dt = \bigcap_{N=1}^{\infty} S_N(F;[a,b]) = \lim_{N\to\infty} S_N(F;[a,b]).$$

Note that if we recursively define

$$\begin{cases} Y_1 = S_1, \\ Y_{k+1} = S_{k+1} \cap Y_k, \qquad k = 1,2,\ldots, \end{cases}$$

where $S_k = S_k(F;[a,b])$, then $\{Y_k\}$ is a nested sequence of intervals that converges to the exact value of the integral $\int_a^b f(t)\,dt$.

Note also that a Lipschitz interval extension F used here has the property that $F(x)$ is a real number for any real number $x \in \Re$. However, for other inclusion monotonic interval functions that are not Lipschitz, the corresponding function $F(x)$ may have interval coefficients even if x is a real number.

Using the interval mathematics introduced above, we have the following particularly important concept, which is crucial to this research.

Let X be a real interval of real-valued random variables in interest, and

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{\frac{-(x-\mu_x)^2}{2\sigma_x^2}\right\}, \qquad x \in X$$

be an ordinary Gaussian density function with known μ_x and $\sigma_x > 0$. Then $f(x)$ has a Lipschitz interval extension, so that the *interval expectation*

$$E\{X\} = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\sigma_x} \exp\left\{\frac{-(x-\mu_x)^2}{2\sigma_x^2}\right\} dx, \qquad x \in X$$

and the *interval variance*

$$V\{X\} = E\{(X - E(X))^2\} = \int_{-\infty}^{\infty} (x-\mu_x)^2 f(x)\,dx = \int_{-\infty}^{\infty} \frac{(x-\mu_x)^2}{\sqrt{2\pi}\sigma_x} \exp\left\{\frac{-(x-\mu_x)^2}{2\sigma_x^2}\right\} dx, \qquad x \in X$$

are both well defined, based on the definite integral defined above with $a \to -\infty$ and $b \to \infty$. Also, with respect to another real interval Y of real-valued random variables, the *conditional interval expectation*

$$E\{X \mid y \in Y\} = \int_{-\infty}^{\infty} x f(x \mid y)\,dx = \int_{-\infty}^{\infty} x \frac{f(x,y)}{f(y)}\,dx = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\sigma_{xy}} \exp\left\{\frac{-(x-\mu_{xy})^2}{2\sigma_{xy}^2}\right\} dx, \qquad x \in X$$

and the *conditional variance*

$$V\{X \mid y \in Y\} = E\{(x-\mu_x)^2 \mid y \in Y\} = \int_{-\infty}^{\infty} (x - E\{x \mid y \in Y\})^2 f(x \mid y)\,dx = \int_{-\infty}^{\infty} (x - E\{x \mid y \in Y\})^2 \frac{f(x,y)}{f(y)}\,dx = \int_{-\infty}^{\infty} \frac{(x - E\{x \mid y \in Y\})^2}{\sqrt{2\pi}\tilde{\sigma}} \times \exp\left\{\frac{-(x-\tilde{\mu})^2}{2\tilde{\sigma}^2}\right\} dx, \qquad x \in X$$

are both well defined, based on the same reasoning and the well-defined interval division operation (note that zero is not contained in the denominator for a Gaussian density interval function), where

$$\tilde{\mu} = \mu_x + \sigma_{xy}^2(y-\mu_y)/\sigma_y^2 \qquad \text{and}$$
$$\tilde{\sigma}^2 = \sigma_x^2 - \sigma_{xy}^2\sigma_{yx}^2/\sigma_y^2$$

with

$$\sigma_{xy}^2 = \sigma_{yx}^2 = E\{XY\} - E\{X\}E\{Y\} = E\{xy\} - E\{x\}E\{y\}, \qquad x \in X.$$

Indeed, it is well known [3] that

$$E\{X \mid y \in Y\} = E\{x\} + \sigma_{xy}^2[y - E\{y\}]/\sigma_y^2, \qquad x \in X$$

and

$$V\{X \mid y \in Y\} = V\{x\} - \sigma_{xy}^2\sigma_{yx}^2/\sigma_y^2, \qquad x \in X.$$

Thus, we have actually verified that all these quantities are well-defined rational interval functions, and so Corollary B can be applied to them.

III. INTERVAL KALMAN FILTER

In this section, we develop the new IKF for linear discrete-time interval systems.

We start with the linear, discrete-time, time-varying, nominal dynamic-observation system

$$\begin{cases} x_{k+1} = A_k x_k + B_k \xi_k, \\ y_k = C_k x_k + \eta_k, \qquad k = 0,1,2,\ldots \end{cases} \tag{1}$$

where $x_k \in \Re^n$ and $y_k \in \Re^m$ are state and output vectors, respectively, with a Gaussian initial state x_0 of known mean $E\{x_0\}$ and covariance $P_0 = V\{x_0\}$, $A_k \in \Re^{n\times n}$, $B_k \in \Re^{n\times p}$, and $C_k \in \Re^{m\times n}$ are known constant matrices, and $\{\xi_k\}$ and $\{\eta_k\}$ are mutually independent zero-mean Gaussian noise sequences, with known covariance matrices $\{Q_k\}$ and $\{R_k\}$, respectively, which are all independent of the initial state, namely,

$$E\{\xi_k,\xi_l\} = Q_k\delta_{kl}, \qquad E\{\eta_k,\eta_l\} = R_k\delta_{kl}$$
$$E\{\xi_k,\eta_l\} = E\{\xi_k,x_0\} = E\{\eta_k,x_0\} = 0$$

for all $k, l = 0,1,2,\ldots$, where $\delta_{kl} = 1$ if $k = l$ and $= 0$ otherwise.

If all the constant matrices $\{A_k,B_k,C_k\}$ are certain, then the Kalman filter for the ordinary system (1) is well known, which gives the optimal estimates $\{\hat{x}_k\}$ of the unknown state vectors $\{x_k\}$ using the observation data $\{y_k\}$ in a very efficient recursive fashion. The optimal estimates are uniquely determined by the conditional expectations [2]

$$\hat{x}_k = E\{x_k \mid y_0,\ldots,y_{k-1}\}. \tag{2}$$

Moreover, the Kalman filtering algorithm provides a recursive scheme for real-time computation of the optimal estimates (2).

1) *Standard Kalman Filter.*

Main Process:

$$\hat{x}_0 = E\{x_0\} \quad \text{(given)}$$
$$\hat{x}_k = A_{k-1}\hat{x}_{k-1} + G_k[y_k - C_kA_{k-1}\hat{x}_{k-1}],$$
$$k = 1,2,\ldots .$$

Coprocess:

$$P_0 = V\{x_0\} \quad \text{(given)}$$
$$M_{k-1} = A_{k-1}P_{k-1}A_{k-1}^\top + B_{k-1}Q_{k-1}B_{k-1}^\top$$
$$G_k = M_{k-1}C_k^\top[C_kM_{k-1}C_k^\top + R_k]^{-1}$$
$$P_k = [I - G_kC_k]M_{k-1}[I - G_kC_k]^\top + G_kR_kG_k^\top$$
$$k = 1,2,\ldots .$$

Given this background, we are now in a position to discuss the following interval system, obtained from system (1) with perturbations in its system matrices:

$$\begin{cases} x_{k+1} = A_k^I x_k + B_k^I\xi_k, \\ y_k = C_k^I x_k + \eta_k, \qquad k = 0,1,2,\ldots \end{cases} \tag{3}$$

where the interval matrices

$$A_k^I = A_k + \Delta A_k = [A_k - |\Delta A_k|, A_k + |\Delta A_k|]$$
$$B_k^I = B_k + \Delta B_k = [B_k - |\Delta B_k|, B_k + |\Delta B_k|]$$
$$C_k^I = C_k + \Delta C_k = [C_k - |\Delta C_k|, C_k + |\Delta C_k|]$$

$k = 0,1,2,\ldots$, in which $\Delta A_k \in \Re^{n\times n}$, $\Delta B_k \in \Re^{n\times p}$ and $\Delta C_k \in \Re^{m\times n}$ are unknown but bounded constant perturbation matrices with the componentwise upper bound denoted by $|\cdot|$, and the other notation and conditions (including the noises) are the same as those defined in system (1).

We show that the optimal estimates for this interval system (2) are given by the interval conditional expectations:

$$\hat{x}_k = E\{x_k \mid y_0,\ldots,y_{k-1}\}, \qquad x_k \in X_k \tag{4a}$$

for some intervals $\{X_k\}$ to be determined later, or simply put,

$$\hat{X}_k = E\{X_k \mid y_0,\ldots,y_{k-1}\}, \qquad k = 1,2,\ldots . \tag{4b}$$

We derive an IKF from (4a) or (4b), for the optimal estimates of the unknown interval state vectors of the system, under the same statistical assumptions as that for the ordinary system (1).

We first note that according to the interval expectation introduced in Section IIB and the standard interval arithmetic operations for interval matrices, the following interval conditional expectation vectors and covariance matrices (as a matrix-version of the results obtained by the end of Section II) are well defined:

$$E\{X \mid y \in Y\} = E\{X\} + R_{xy}[R_{yy}]^{-1}R_{yx}[y - E\{y\}] \tag{5}$$

and

$$V\{X \mid y \in Y\} = V\{X\} - R_{xy}[R_{yy}]^{-1}R_{yx} \tag{6}$$

where $R_{yy} = V\{yy^\top\} > 0$, $R_{yx} = R_{xy}^\top$, and

$$R_{xy} = V\{xy^\top \mid y \in Y\}, \qquad x \in X.$$

Therefore, the interval conditional expectations (4) are well defined for the interval system (3) and hence are readily to use. Based on these results, we can now establish the following new algorithm.

2) *Interval Kalman Filter.*

Main Process:

$$\hat{x}_0^I = E\{x_0^I\} \quad \text{(given)}$$
$$\hat{x}_k^I = A_{k-1}^I\hat{x}_{k-1}^I + G_k^I[y_k^I - C_k^IA_{k-1}^I\hat{x}_{k-1}^I],$$
$$k = 1,2,\ldots .$$

Coprocess:

$$P_0^I = V\{x_0^I\} \quad \text{(given)}$$
$$M_{k-1}^I = A_{k-1}^IP_{k-1}^I[A_{k-1}^I]^\top + B_{k-1}^IQ_{k-1}[B_{k-1}^I]^\top$$
$$G_k^I = M_{k-1}^I[C_k^I]^\top[[C_k^I]M_{k-1}^I[C_k^I]^\top + R_k]^{-1}$$
$$P_k^I = [I - G_k^IC_k^I]M_{k-1}^I[I - G_k^IC_k^I]^\top + [G_k^I]R_k[G_k^I]^\top \qquad k = 1,2,\ldots .$$

Since all the interval arithmetic and conditional expectation and covariance have been well defined for matrices (and hence vectors and numbers), the above IKF algorithm can be verified by imitating the

derivation of the SKF on a step-by-step basis [2], and so its proof is omitted.

It should be noted that by theory this IKF algorithm is optimal for the interval system (3), in the same sense as the SKF, namely, in the statistical sense of the conditional expectation (4), since no approximation has been used in its derivation. The filtering result produced by the IKF is a sequence of interval estimates $\{\hat{X}_k\}$ (see (4b)) that contains all possible optimal estimates $\{\hat{x}_k\}$ of the state vectors $\{x_k\}$ that the interval system (3) may generate. Hence, the filtering result produced by the IKF is generally conservative and expanding rapidly, as expected.

It should also be remarked that just like the random vector (observation data) y_k in the SKF scheme, the interval data vector y_k^I shown in the IKF above is an uncertain interval vector before its realization (i.e., before the data actually being obtained), but is an ordinary constant vector after it has been realized and obtained. This should avoid possible confusion in implementing the new IKF algorithm.

Regarding the computation of the IKF, we remark that similar to the SKF where the matrix inversion in the algorithm may have the singularity problem and is generally time consuming, in the IKF scheme the interval matrix inversion (although well defined) may also have the singularity problem (if it contains zero in some interval entries), which often times causes divergence in implementations. A general analysis of this convergence issue is rather difficult, even for the SKF, as is well known, and so not investigated here.

Some efficient computational schemes for the interval matrix inversion needed by the IKF are available in the literature, including the Hansen algorithm [1] and the iteration method for matrix inverse inclusion [1]. A suboptimal IKF scheme for real-time implementation of the optimal IKF is suggested in the next section, which replaces the interval matrix inversion by an ordinary matrix inversion, using the worst case matrix inversion instead of the optimal interval matrix inversion.

IV. SUBOPTIMAL INTERVAL KALMAN FILTER

To improve the computational efficiency, different approximations are quite possible. In this section, we suggest a suboptimal interval Kalman filtering scheme, by replacing its interval matrix inversion with its worst case inversion, while keeping everything else unchanged.

Let

$$C_k^I = C_k + \Delta C_k \qquad \text{and} \qquad M_{k-1}^I = M_{k-1} + \Delta M_{k-1}$$

where C_k = center point of C_k^I and M_{k-1} = center point of M_{k-1}^I are the nominal values of the interval matrices, as defined in (3). Write

$$\begin{aligned}[[C_k^I]M_{k-1}^I[C_k^I]^\top + R_k]^{-1} &= [[C_k + \Delta C_k][M_{k-1} + \Delta M_{k-1}][C_k + \Delta C_k]^\top + R_k]^{-1} \\ &= [C_k M_{k-1} C_k^\top + \Delta R_k]^{-1}\end{aligned}$$

where

$$\begin{aligned}\Delta R_k = {} & C_k M_{k-1}[\Delta C_k]^\top + C_k[\Delta M_{k-1}]C_k^\top \\ & + C_k[\Delta M_{k-1}][\Delta C_k]^\top \\ & + [\Delta C_k]M_{k-1}C_k^\top + [\Delta C_k]M_{k-1}[\Delta C_k]^\top \\ & + [\Delta C_k][\Delta M_{k-1}]C_k^\top \\ & + [\Delta C_k][\Delta M_{k-1}][\Delta C_k]^\top + R_k.\end{aligned}$$

Finally, in the IKF, replace ΔR_k by its upper bound matrix, denoted by $|\Delta R_k|$, which consists of all the upper bounds of the interval elements of ΔR_k. Namely, denoting

$$\Delta R_k = [[-r_k(i,j), r_k(i,j)]]$$

with $r_k(i,j) \geq 0$, we let

$$|\Delta R_k| = [r_k(i,j)]. \tag{7}$$

We should note that this $|\Delta R_k|$ is a regular (noninterval matrix), so that when we use the regular inverse matrix $[C_k M_{k-1} C_k^\top + |\Delta R_k|]^{-1}$ to replace the interval matrix inverse $[[C_k^I]M_{k-1}^I[C_k^I]^\top + R_k]^{-1}$, the matrix inversion becomes much easier. More importantly, we should note that when the perturbation matrix $\Delta C_k = 0$ in (7), meaning that the measurement equation in system (1) or (3) is as accurate as the SKF model, then we have $|\Delta R_k| = R_k$. However, any of such approximations will lose some possible solutions of the optimal filtering, which we should be aware of in applications.

Thus, using the replacement of ΔR_k by $|\Delta R_k|$, we arrive at the following *suboptimal* IKF scheme.

3) *Suboptimal Interval Kalman Filter.*

Main Process:

$$\begin{aligned}\hat{x}_0^I &= E\{x_0^I\} \quad \text{(given)} \\ \hat{x}_k^I &= A_{k-1}^I \hat{x}_{k-1}^I + G_k^I[y_k^I - C_k^I A_{k-1}^I \hat{x}_{k-1}^I], \\ & \quad k = 1,2,\ldots.\end{aligned}$$

Coprocess:

$$\begin{aligned}P_0^I &= V\{x_0^I\} \quad \text{(given)} \\ M_{k-1}^I &= A_{k-1}^I P_{k-1}^I [A_{k-1}^I]^\top + B_{k-1}^I Q_{k-1}[B_{k-1}^I]^\top \\ G_k^I &= M_{k-1}^I [C_k^I]^\top [C_k M_{k-1} C_k^\top + |\Delta R_k|]^{-1} \\ P_k^I &= [I - G_k^I C_k^I] M_{k-1}^I [I - G_k^I C_k^I]^\top \\ & \quad + [G_k^I] R_k [G_k^I]^\top \qquad k = 1,2,\ldots.\end{aligned}$$

We finally remark that the worst case matrix $|\Delta R_k|$ given in (7) contains the largest possible perturbations and is in some sense the "best" matrix

that yields a numerically stable inverse. Another possible approximation is, if ΔC_k is small, to simply use $|\Delta R_k| \approx R_k$ (see (7)). For some specific systems such as the radar tracking system to be discussed and simulated in the next section, special techniques are also possible to improve the speed and/or accuracy of a suboptimal interval filtering.

V. COMPUTER SIMULATIONS AND COMPARISONS

In this section, we show some computer simulations with two different types of comparisons: a comparison of the IKF with the SKF for a simplified radar tracking system; a comparison of the IKF with some other existing comparable methods for the same examples that those approaches used.

A. Comparison with Standard Kalman Filter

To simplify the presentation, we only show a very simple, well-known tracking system [2, 3] of the form

$$\begin{cases} x_{k+1} = \begin{bmatrix} 1 & h^I \\ 0 & 1 \end{bmatrix} x_k + \xi_k \\ y_k = [1 \quad 0] x_k + \eta_k \end{cases} \tag{8}$$

where basic assumptions are as stated in system (1), with

$$h^I = [h - \Delta h, h + \Delta h] = [0.01 - 0.001, 0.01 + 0.001] = [0.009, 0.011]$$

in which the modeling error Δh was taken to be 10% of the nominal value of $h = 0.01$, and the given data are

$$E\{x_0\} = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$V\{x_0\} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}$$

$$Q_k = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 0.1 \end{bmatrix},$$

$$R_k = r = 0.1.$$

For this model, using the IKF described in Section III, we have

$$M^I_{k-1} = \begin{bmatrix} h^I[2P^I_{k-1}(1,0) + h^I P^I_{k-1}(1,1)] + P^I_{k-1}(0,0) + q & P^I_{k-1}(0,1) + h^I P^I_{k-1}(1,1) \\ P^I_{k-1}(1,0) + h^I P^I_{k-1}(1,1) & P^I_{k-1}(1,1) + q \end{bmatrix} := \begin{bmatrix} M^I_{k-1}(0,0) & M^I_{k-1}(0,1) \\ M^I_{k-1}(1,0) & M^I_{k-1}(1,1) \end{bmatrix}$$

$$G^I_k = \begin{bmatrix} 1 - r/(M^I_{00} + r) \\ M^I_{10}/(M^I_{00} + r) \end{bmatrix} := \begin{bmatrix} G^I_{k,1} \\ G^I_{k,2} \end{bmatrix}$$

$$P^I_k = \begin{bmatrix} rG^I_{k,1} & rG^I_{k,2} \\ rG^I_{k,2} & q + [P^I_{k-1}(1,1)[P^I_{k-1}(0,0) + q + r] - [P^I_{k-1}(0,1)]^2]/(M^I_{k-1}(0,0) + r) \end{bmatrix} := \begin{bmatrix} P^I_k(0,0) & P^I_k(0,1) \\ P^I_k(1,0) & P^I_k(1,1) \end{bmatrix}.$$

In the derivation of this algorithm, when we have interval division of the type X^I/X^I where X^I does not contain zero, we first examine its corresponding regular (noninterval) functions and operations to obtain $x/x = 1$, and then return to the interval setting. Thus, the interval division X^I/X^I will not actually be encountered. Symbolically, we may directly write $X^I/X^I = 1$ for an interval X^I not containing zero in the interval setting, which is indeed a convention in interval calculations.

Moreover, we note that the matrices M^I_{k-1} and P^I_k are both symmetrical, so that $M^I_{k-1}(0,1) = M^I_{k-1}(1,0)$ and $P^I_{k-1}(0,1) = P^I_{k-1}(1,0)$.

Consequently, we obtain

$$\begin{bmatrix} \hat{x}^I_{k,1} \\ \hat{x}^I_{k,2} \end{bmatrix} = \begin{bmatrix} [r(\hat{x}^I_{k-1,1} + h^I \hat{x}^I_{k-1,2}) + M^I_{k-1}(0,0) y_k]/M^I_{k-1}(0,0) \\ \hat{x}^I_{k-1,2} + G^I_{k,1}(y_k - \hat{x}^I_{k-1,1} - h^I \hat{x}^I_{k-1,2}) \end{bmatrix}.$$

The simulation results for $\hat{x}_{k,1}$ of this IKF versus the SKF, where the latter uses the nominal value of h^I, are shown in Figs. 1 and 2. From the figures we can see that the IKF does give the upper and lower boundaries of the SKF estimated curve. As the iterations continue, the two boundaries are expanding, due to the nature of the given interval systems.

B. Comparisons with Some Other Methods

In this subsection, we briefly discuss comparisons of our IKF algorithm with two existing, comparable approaches [15, 18], using the same examples that were used therein. There are some other approaches as mentioned in the Introduction. However, since the conditions and assumptions are very different, a comparison with any of those methods is indeed impossible.

1) *Comparison with Method of* [18]: The approach given in [18] can only be applied to time-invariant systems. To compare with the example studied in [18], we use our notation and rewrite their

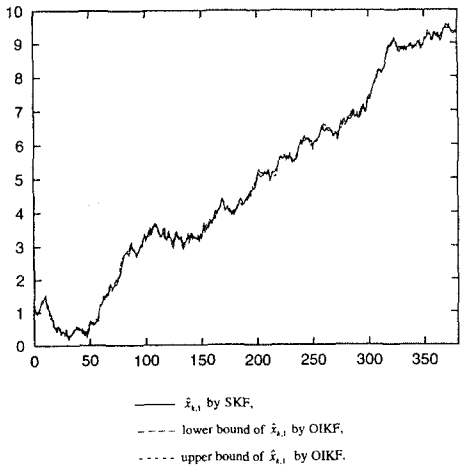


Fig. 1. Simulation results for model (8): optimal IKF algorithm. ——— $\hat{x}_{k,1}$ by SKF, – – – – lower bound of $\hat{x}_{k,1}$ by OIKF, - - - - - upper bound of $\hat{x}_{k,1}$ by OIKF.

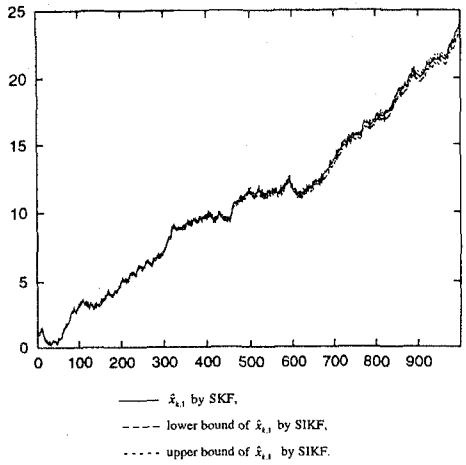


Fig. 2. Simulation results for model (8): suboptimal IKF algorithm. ——— $\hat{x}_{k,1}$ by SKF, – – – – lower bound of $\hat{x}_{k,1}$ by SIKF, - - - - - upper bound of $\hat{x}_{k,1}$ by SIKF.

example as

$$A = \begin{bmatrix} 0.4 & 0.1 \\ -0.1 & 0.2 \end{bmatrix},$$

$$\Delta A = \begin{bmatrix} [-0.1, 0.1] & [-0.15, 0.15] \\ 0 & [-0.25, 0.25] \end{bmatrix}$$

$$C = [0 \quad 1], \qquad \Delta C = [0 \quad [-0.1, 0.1]], \qquad B \equiv I.$$

In [18], it is allowed that the noise covariance matrices Q and R have perturbations ΔQ and ΔR, respectively. For this example,

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \qquad \Delta Q = \begin{bmatrix} [-2, 2] & 0 \\ 0 & [-2, 2] \end{bmatrix}$$

$$R = 1, \qquad \Delta R = [-0.9, 1.1].$$

Then, according to the (first) method that they derived, the following Riccati equation has to be solved for a positive definite solution:

$$S = Q + A[S - SC^{\top}[CSC^{\top} + R]^{-1}CS]A^{\top}$$

which, for this example, is

$$S = \begin{bmatrix} 11.9081 & -0.4602 \\ -0.4602 & 10.1570 \end{bmatrix}.$$

The mean square estimation error according to the method of [18] is given, after solving for other two Riccati equations, by

$$16.0362 \leq \lim_{k \to \infty} E\{[x_k - \hat{x}_k]^{\top}[x_k - \hat{x}_k]\} \leq 33.0225.$$

By using our optimal IKF and suboptimal IKF algorithms derived in this work, also compared with the SKF scheme, using the same model and the same data described above, we obtained the filtering results as shown in Figs. 3 and 4 (where, to save space only the second components of $\hat{x}_k$ are shown). It is clear that the results are consistent with that shown in Figs. 1 and 2, since both the optimal and suboptimal IKF algorithms gave the upper and lower boundaries for the estimations. The upper and lower estimation error bounds we obtained for the suboptimal IKF are

$$16.344253 \leq \lim_{k \to \infty} E\{[x_k - \hat{x}_k]^{\top}[x_k - \hat{x}_k]\} \leq 27.929521$$

which are very close to, indeed slightly better than, that obtained in [18] shown above. The corresponding result we obtained using the optimal IKF is

$$4.5 \leq \lim_{k \to \infty} E\{[x_k - \hat{x}_k]^{\top}[x_k - \hat{x}_k]\} \leq 79.579$$

which includes all possible optimal solution bounds.

Here, we should remark that our algorithm is applicable to time-varying systems as well. Besides, since the statistics such as the white Gaussian properties of the noises have been assumed, which is

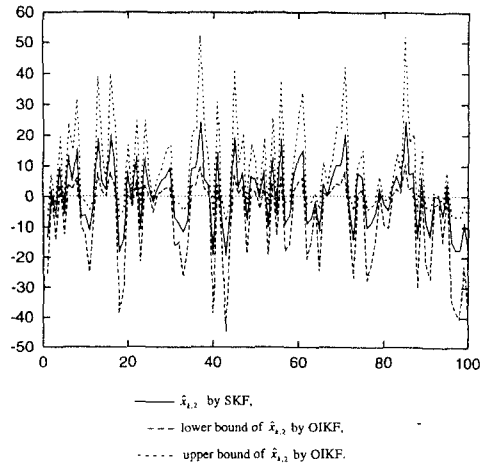


Fig. 3. Simulation results for model of [18]: ——— $\hat{x}_{k,2}$ by SKF, – – – – lower bound of $\hat{x}_{k,2}$ by OIKF, - - - - - upper bound of $\hat{x}_{k,2}$ by OIKF.

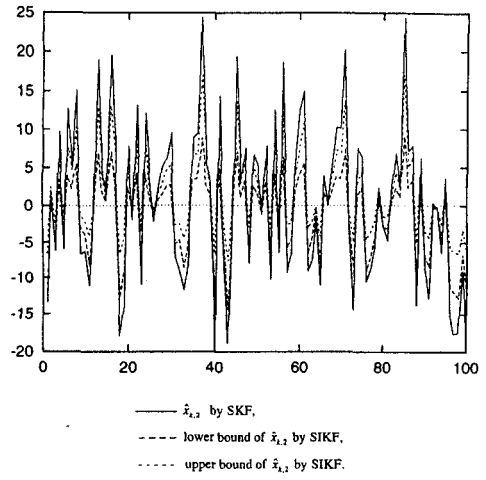


Fig. 4. Simulation results for model of [18]: ——— $\hat{x}_{k,2}$ by SKF, – – – – lower bound of $\hat{x}_{k,2}$ by SIKF, - - - - - upper bound of $\hat{x}_{k,2}$ by SIKF.

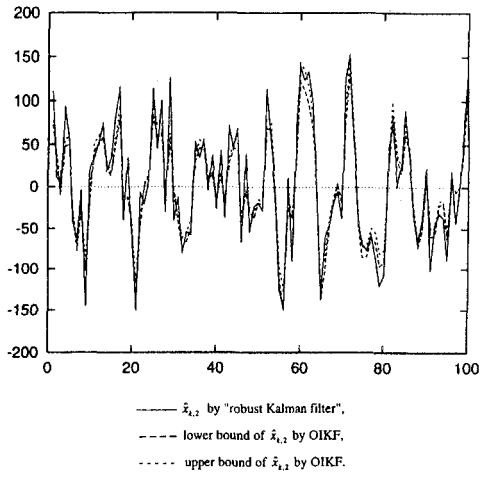


Fig. 5. Simulation results for model of [15]: ——— $\hat{x}_{k,2}$ by "robust Kalman filter," – – – – lower bound of $\hat{x}_{k,2}$ by OIKF, - - - - - upper bound of $\hat{x}_{k,2}$ by OIKF.

the same as the SKF requirement. We believe that it is more reasonable not to further consider the noise covariance perturbations $\{\Delta Q_k\}$ and $\{\Delta R_k\}$.

2) *Comparison with Method of* [15]: The system model considered in [15] is closer to our model, where time-varying model variations $\{\Delta A_k\}$ and $\{\Delta C_k\}$ are allowed. The example discussed in [15] is slightly modified as follows:

$$A = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix}, \qquad \Delta A = \begin{bmatrix} 0 & 0 \\ 0 & [-0.3, 0.3] \end{bmatrix}$$

$$C = [-100 \quad 10], \qquad \Delta C = 0, \qquad B \equiv I$$

and the noises are zero-mean white Gaussian with unity covariance.

According to [15], we first need to write

$$\begin{bmatrix} \Delta A_k \\ \Delta C_k \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F_k E$$

for a matrix F_k satisfying $F_k^\top F_k \le I$, where H_1, H_2, and E are known constant matrices of appropriate dimensions. Then, according to [15], for any fixed small $\nu > 0$ there is a sufficiently small $\varepsilon > 0$ such that the Riccati equation (for this example)

$$A^\top P A - P + A^\top P (I - P)^{-1} P A + \varepsilon (E^\top E + \nu I) = 0$$

has a stabilizing solution P, and the upper bound M for the estimation errors is defined by

$$E\{[x_k - \hat{x}_{k,k-1}]^\top [x_k - \hat{x}_{k,k-1}]\} \le M.$$

Applying the optimal IKF, the lower and upper bounds for M were obtained as [3.94890, 4.53796] in our simulation.

We finally show the filtering results of the optimal and suboptimal IKF algorithm versus the robust filtering scheme of [15] for this example in Fig. 5, where the behavior of the IKF filtering curves is consistent with the examples shown above.

VI. CONCLUSIONS

The classical Kalman filtering technique has been extended to interval linear systems with the same statistical assumptions on noise in this work, for which the classical algorithm is no longer applicable. The new interval Kalman filtering scheme has the same optimality and recursive structure as the classical Kalman filtering algorithm, using no additional analysis or computation from such as H^∞-mathematics. A suboptimal IKF has also been suggested for the purpose of real-time implementation. Computer simulations have shown that the new interval Kalman filtering algorithm is consistent with the classical Kalman filtering scheme, and is also consistent with (or better than) some existing robust Kalman filtering methods.

REFERENCES

[1] Alefeld, G., and Herzberger, J. (1983)
Introduction to Interval Computations.
New York: Academic Press, 1983.

[2] Chen, G., Chen, G., and Hsu, S. H. (1995)
Linear Stochastic Control Systems.
Boca Raton, FL: CRC Press, 1995.

[3] Chui, C. K., and Chen, G. (1991)
Kalman Filtering with Real-Time Applications (2nd ed.).
New York: Springer-Verlag, 1991.

[4] Haddad, W. M., Berstein, D. S., and Mustafa, D. (1991)
Mixed-norm H_2/H_∞ regulation and estimation: The discrete-time case.
System Control Lett., **16** (1991), 235–247.

[5] Hong, L. (1993)
Distributed filtering using set models for systems with non-Gaussian noise.
In G. Chen (Ed.), *Approximate Kalman Filtering.*
Singapore: World Scientific Publishing Co., Ltd., 1993, 161–176.

[6] Jaulin, L., and Walter, E. (1993)
Set inversion via interval analysis for nonlinear bounded-error estimation.
Automatica, **29** (1993), 1053–1064.

[7] Luo, J. S., and Johnson, A. (1992)
Stability robustness of the discrete-time LQG system under plant perturbation and noise uncertainty.
International Journal of Control, **55** (1992), 1491–1502.

[8] Martin, C. J., and Mintz, M. (1993)
Robust filtering and prediction for linear systems with uncertain dynamics: A game theoretic approach.
IEEE Transactions on Automatic Control, **28** (1993), 888–896.

[9] Milanese, M., and Belforte, G. (1982)
Estimation theory and uncertainty intervals evaluation in presence of unknown-but-bounded errors: Linear families of models and estimators.
IEEE Transactions on Automatic Control, **AC-27** (1982), 408–414.

[10] Moore, R. E. (1979)
Methods and Applications of Interval Analysis.
Philadelphia: SIAM Press, 1979.

[11] Morrell, D., and Stirling, W. C. (1993)
Set-valued and distributed Kalman filtering.
In G. Chen (Ed.), *Approximate Kalman Filtering*.
Singapore: World Scientific Publishing Co., Ltd., 1993, 139–160.

[12] Nagpal, K. M., and Khargonekar, P. P. (1991)
Filtering and smoothing in an H_∞ setting.
IEEE Transactions on Automatic Control, **36** (1991), 152–166.

[13] Poznyak, A. S., and Cordero, A. O. (1994)
Suboptimal robust filtering of states of finite dimensional linear systems with time-varying parameters under nonrandom disturbances.
In *Proceedings of the 33rd Conference on Decision Control*, 1994, 3931–3936.

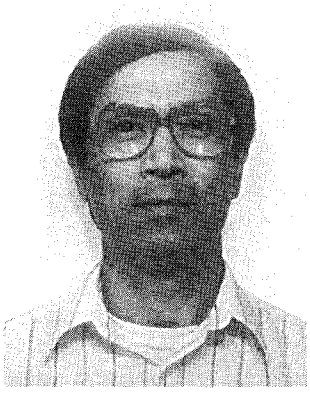
[14] Shaked, U., and Theodor, Y. (1992)
H_∞-optimal estimation: A tutorial.
In *Proceedings of the 31st Conference on Decision Control*, Tucson, AZ, Dec. 1992, 2278–2286.

[15] Xie, L., Soh, Y. C., and de Souza, C. E. (1994)
Robust Kalman filtering for uncertain discrete-time systems.
IEEE Transactions on Automatic Control, **39** (1994), 1310–1314.

[16] Xie, L., de Souza, C. E., and Fu, M. (1991)
H_∞ estimation for discrete-time linear uncertain systems.
International Journal of Robust Nonlinear Control, **1** (1991), 111–123.

[17] Xie, L., de Souza, C. E., and Wang, Y. (1994)
H_∞ filter design for discrete-time uncertain nonlinear systems.
In *Proceedings of the 33rd Conference on Decision Control*, 1994, 3937–3942.

[18] Zhang, X. F., Heemink, A. W., and van Eijkeren, J. C. H. (1995)
Performance robustness analysis of Kalman filter for linear discrete-time systems under plant and noise uncertainty.
International Journal of System Science, **26** (1995), 257–275.



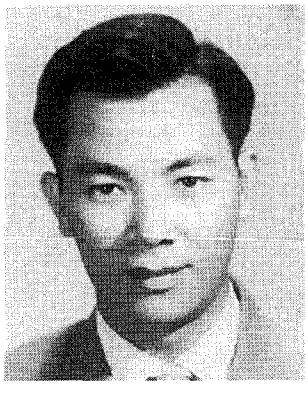
Guanrong Chen (M'89—SM'92) received the M.S. degree in computer science from Sun Yatsen (Zhongshan) University in 1981 and the Ph.D. degree in applied mathematics from Texas A&M University, College Station, in 1987.

He was a visiting Assistant Professor at Rice University from 1987 to 1990, in both the Department of Electrical and Computer Engineering and the Department of Mathematical Sciences, and joined the faculty of the Department of Electrical Engineering, University of Houston in 1990.

Dr. Chen is the (co)author of six research monographs and textbooks entitled *Kalman Filtering with Real-Time Applications* (1987), *Linear Systems and Optimal Control* (1989), *Signal Processing and Systems Theory: Selected Topics* (1992), *Nonlinear Feedback Control Systems: An Operator Theory Approach* (1993), *Approximate Kalman Filtering* (1993), and *Linear Stochastic Control Systems* (1995), in addition to 70 some journal papers, mainly in the field of nonlinear systems control. He is a former Associate Editor of the *IEEE Transactions on Circuits and Systems* and a current Associate Editor for the Chinese *Journal of Control Theory and Applications.*

Jianrong Wang born in Fujian, China, in January 1954. She received the BSEE degree from the Northwestern Telecommunication Engineering Institute (now, the XIAN University of Electronics), China, and the MSEE degree from the University of Houston, Houston, TX.

She has more than ten years of experience in telecommunication and a few years of experience in systems and control laboratories. Her research interest is in the area of communication, signals, and systems engineering.



Leang-San Shieh (S'69—M'70—SM'92) received his B.S. degree from the National Taiwan University, Taiwan, in 1958, and the M.S. and Ph.D. degrees from the University of Houston, Houston, TX, in 1968 and 1970, respectively, all in electrical engineering.

He joined the Department of Electrical and Computer Engineering at the University of Houston in 1971 as an Assistant Professor and was promoted to Associate Professor in 1974, and Professor in 1978. Since 1988, he has been the Director of the Computer and System Engineering Program.

Dr. Shieh is a member of IEEE and AIAA, and a registered Professional Engineer in the State of Texas. He was the recipient of the 1973 College Teaching Excellence Award and the 1988 College Senior Faculty Research Excellence Award from the Cullen College of Engineering, University of Houston, and the 1976 University Teaching Excellence Award from the University of Houston. Further, he received the Honor of Merit from Instituto Universitario Politecnico, Republic of Venezuela in 1978. His fields of interest are digital control, optimal control, self-tuning control, and hybrid control of uncertain systems. He has published more than 170 articles in various scientific journals. He co-authored a research monograph, *An Algebraic Approach to Structural Analysis and Design of Multivariable Control Systems* (Springer, New York) in 1988.