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Reliable Optimal Production Control with Cobb-Douglas Model

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Abstract. Production is the most fundamental activity in our economy. In this paper, a Cobb-Douglas production function is used as the mathematical model to describe the relationship among production, labor and capital. Two reliable production optimal control problems are studied. Algorithms to find dynamic optimal control intervals are provided with interval parameter presentations and interval computations.

1. Optimal Production Control with Cobb-Douglas Model: Traditional (Non-Interval) Case

Production. Every day, various products are produced to meet different demands in our society. Production is truly one of the most fundamental activities in our economy.

Every producing firm wants to maximize its profits:

- In an *equilibrium* market, where the amount produced is more or less fixed (by the demand and by the firm's market share), in order to maximize profits, the firm needs to minimize production costs.
- In a *seller's* market, in which the supply of a product is smaller than the demand for it, maximizing profits means producing the maximum amount within the available production costs.

Production function. In both cases, to optimize production, we must know how the output Q depends on the production costs. The dependence of Q on controllable parameters is called a *production function*.

One of the most widely used production functions was proposed by Cobb and Douglas and has the following form (see, e.g., [3]):

$$Q = A \cdot L^{\alpha} \cdot K^{\beta}, \tag{1}$$

where L is *labor* (measured in certain units), K is *capital*, and A, α, and β are (constant) parameters; these parameters depend on the firm, on the produced unit, etc., and have to be determined experimentally.

Comment. In many real-life situations, $\alpha + \beta = 1$. This equality has a simple economic interpretation: if we increase *both* labor *and* capital twofold, we will thus produce twice as many units of the product. This additional assumption makes the corresponding formulas simpler.

Cost function. The production cost C consists of the cost of labor + the cost of capital:

$$C = a \cdot L + b \cdot K, \tag{2}$$

where a is the cost of a unit of labor, and b is the cost of acquiring a unit of capital.

Depending on the market, we have one of the following two optimization problems:

Optimization problem for the equilibrium market:
Given: A, α, β, a, b, and Q.
Minimize: $C = a \cdot L + b \cdot K$.
Subject to: $Q = A \cdot L^\alpha \cdot K^\beta$.

Optimization problem for the seller's market:
Given: A, α, β, a, b, and C.
Maximize: $Q = A \cdot L^\alpha \cdot K^\beta$.
Subject to: $C = a \cdot L + b \cdot K$.

How we can solve these problems. Both conditional optimization problems can be solved by using the Lagrange multiplier method (see, e.g., [7]). In both cases, we can even get an explicit expression for the solution.

Lagrange Multiplier method reduces both problems to the same (unconditional) optimization problem:

$$A \cdot L^\alpha \cdot K^\beta - \lambda \cdot (a \cdot L + b \cdot K) \to \max,$$

where λ is the (unknown) Lagrange multiplier. We can simplify the problem by dividing the objective function by a (positive) constant A:

$$L^\alpha \cdot K^\beta - \mu \cdot (a \cdot L + b \cdot K) \to \max,$$

where we denoted $\mu = \lambda / A$. Differentiating the new objective function w.r.t. each of the variables L and K, and equating these derivatives to 0, we get the following equations:

$$\alpha \cdot L^{\alpha-1} \cdot K^\beta = \mu \cdot a; \tag{3}$$

$$\beta \cdot L^\alpha \cdot K^{\beta-1} = \mu \cdot b. \tag{4}$$

To eliminate the auxiliary parameter μ from this system, we divide (3) by (4), thus getting:

$$\frac{\alpha}{\beta} \cdot \frac{K}{L} = \frac{a}{b} \tag{5}$$

and

$$K = L \cdot \frac{a}{b} \cdot \frac{\beta}{\alpha}. \tag{6}$$

Substituting this expression for K in terms of L into the corresponding condition, we get, in both cases, an (easily solvable) equation from which we can determine the optimal amount of labor L. From this expression, using (6), we can get an explicit expression for the optimal amount of capital K:

Optimal control for the equilibrium market:

$$L = \left(\frac{Q}{A}\right)^{1/(\alpha+\beta)} \cdot \left(\frac{\alpha}{\beta} \cdot \frac{b}{a}\right)^{\beta/(\alpha+\beta)}; \tag{7}$$

$$K = \left(\frac{Q}{A}\right)^{1/(\alpha+\beta)} \cdot \left(\frac{\beta}{\alpha} \cdot \frac{a}{b}\right)^{\alpha/(\alpha+\beta)}. \tag{8}$$

Comment. As we have mentioned, when $\alpha + \beta = 1$, these equations take an even simpler form:

$$L = \left(\frac{Q}{A}\right) \cdot \left(\frac{\alpha}{\beta} \cdot \frac{b}{a}\right)^{\beta}; \tag{7a}$$

$$K = \left(\frac{Q}{A}\right) \cdot \left(\frac{\beta}{\alpha} \cdot \frac{a}{b}\right)^{\alpha}. \tag{8a}$$

Optimal control for the seller's market:

$$L = \frac{\alpha}{\alpha+\beta} \cdot \frac{C}{a}; \tag{9}$$

$$K = \frac{\beta}{\alpha+\beta} \cdot \frac{C}{b}. \tag{10}$$

These formulas can also be simplified if $\alpha + \beta = 1$:

$$L = \frac{\alpha \cdot C}{a}; \tag{9a}$$

$$K = \frac{\beta \cdot C}{b}. \tag{10a}$$

2. A More Realistic (Interval) Case

Why intervals? If we know all the parameters *exactly*, then it makes sense to try to follow recommendations (7)–(8) (or (9)–(10)) precisely.

In real life, the parameters A, α, and β of the Cobb-Douglas model, the per-unit costs a and b, the required production level Q and the available total cost C are only *approximately* known. At best, we know *intervals* of possible values of these parameters: $\mathbf{A} = [\underline{A}, \overline{A}]$, $\alpha^* = [\underline{\alpha}, \overline{\alpha}]$, $\beta^* = [\underline{\beta}, \overline{\beta}]$, $\mathbf{a} = [\underline{a}, \overline{a}]$, $\mathbf{b} = [\underline{b}, \overline{b}]$, and $\mathbf{Q} = [\underline{Q}, \overline{Q}]$ (or $\mathbf{C} = [\underline{C}, \overline{C}]$).

Interval control. Different values of the parameters $A, \ldots,$ from the given intervals $\mathbf{A}, \ldots,$ lead to different optimal values of labor and capital. In such situations, it is reasonable to supply the decision-maker not with a pair of *numbers* (L, K), but with *intervals* $\mathbf{K} = [\underline{K}, \overline{K}]$ and $\mathbf{L} = [\underline{L}, \overline{L}]$ of possibly optimal values of K and L.

The decision maker will then use his experience and intuition to select the values K and L from these intervals.

To estimate the desired intervals $\mathbf{K}$ and $\mathbf{L}$, we suggest to use (naive) *interval computations* [1], [5], [6], i.e., to replace each operation with real numbers by the corresponding operation with intervals. To implement interval computation on a computer, we used the INTLIB library [2].

Optimal interval control for the seller's market. Formulas (9a) and (10a) that correspond to the seller's market can be implemented in a straightforward way:

$$\mathbf{L} = \frac{\alpha^* \cdot \mathbf{C}}{\mathbf{a}}; \tag{9a$'$}$$

$$\mathbf{K} = \frac{\beta^* \cdot \mathbf{C}}{\mathbf{b}}. \tag{10a$'$}$$

In these formulas, every variable occurs only once and therefore, the results of interval computations coincides with the exact intervals of possible values of K and L (see, e.g., [1], [6]).

In the formulas (9) and (10), the variable α occurs both in the numerator and in the denominator; this can cause an overestimation. To avoid it, we rewrite $\alpha / (\alpha + \beta)$ as $1 / (1 + \beta / \alpha)$. If we apply naive interval computations to thus re-written formula, we get the following expressions:

$$\mathbf{L} = \frac{1}{1 + (\beta^* / \alpha^*)} \cdot \frac{\mathbf{C}}{\mathbf{a}}; \tag{9$'$}$$

$$\mathbf{K} = \frac{1}{1 + (\alpha^* / \beta^*)} \cdot \frac{\mathbf{C}}{\mathbf{b}}. \tag{10$'$}$$

In these formulas, every variable occurs only once and therefore, the results of interval computations also coincides with the exact intervals of possible values of K and L.

Optimal interval control for the equilibrium market. The implementation of the formulas (7), (8), (7a), and (8a) requires some extra work: Namely, since INTLIB does not contain the interval function "to-the-power" x^y, we first have to represent expressions of this type as $\exp(y \cdot \ln(x))$:

$$L = \exp\left\{ \frac{1}{\alpha + \beta} \cdot \big(\ln(Q) - \ln(A)\big) + \frac{\beta}{\alpha + \beta} \cdot \big(\ln(\alpha) - \ln(\beta) + \ln(b) - \ln(a)\big) \right\}; \tag{7$'$}$$

$$K = \exp\left\{\frac{1}{\alpha+\beta}\cdot\big(\ln(Q)-\ln(A)\big)\right.$$
$$\left.+\frac{\alpha}{\alpha+\beta}\cdot\big(\ln(\beta)-\ln(\alpha)+\ln(a)-\ln(b)\big)\right\}; \quad (8')$$

and only then apply interval computations:

$$\mathbf{L} = \exp\left\{\frac{1}{\alpha^*+\beta^*}\cdot\big(\ln(\mathbf{Q})-\ln(\mathbf{A})\big)\right.$$
$$\left.+\frac{1}{(\alpha^*/\beta^*)+1}\cdot\big(\ln(\alpha^*)-\ln(\beta^*)+\ln(\mathbf{b})-\ln(\mathbf{a})\big)\right\}; \quad (7')$$

$$\mathbf{K} = \exp\left\{\frac{1}{\alpha^*+\beta^*}\cdot\big(\ln(\mathbf{Q})-\ln(\mathbf{A})\big)\right.$$
$$\left.+\frac{1}{1+(\beta^*/\alpha^*)}\cdot\big(\ln(\beta^*)-\ln(\alpha^*)+\ln(\mathbf{a})-\ln(\mathbf{b})\big)\right\}. \quad (8')$$

These formulas can also be simplified if $\alpha+\beta=1$:

$$\mathbf{L} = \frac{\mathbf{Q}}{\mathbf{A}}\cdot\exp\left\{\beta^*\cdot\big(\ln(\alpha^*)-\ln(\beta^*)+\ln(\mathbf{b})-\ln(\mathbf{a})\big)\right\}; \quad (7a')$$

$$\mathbf{K} = \frac{\mathbf{Q}}{\mathbf{A}}\cdot\exp\left\{\alpha^*\cdot\big(\ln(\beta^*)-\ln(\alpha^*)+\ln(\mathbf{a})-\ln(\mathbf{b})\big)\right\}. \quad (8a')$$

3. Optimal Control in a Dynamically Changing Environment

Formulation of the problem. In real life, decisions are made in a dynamically changing environment. For example, innovations can make the production process less labor-intensive; capital can be become more easily available, etc. As a result, the parameters A, α, β, etc., can change. If we have been using some values L and K from the optimal intervals, and the situation has changed, then we want to check whether these values L and K are still *possibly optimal* (i.e., whether they still belong to the new optimal intervals $\mathbf{L}$ and $\mathbf{K}$).

- If both values L and K are still possibly optimal, we do not need to change anything in our production.
- If at least one of the values L and K is outside the corresponding new optimal interval, then we need to choose new values for L and K.

Seller's market: naive interval computations are sufficient. For the seller's market, the above formula (9′), (10′), (9a′), and (10a′) give the *exact* interval of possibly optimal values. So, for the seller's market, to check whether the old parameters L_{old} and K_{old} are still possibly optimal, it is sufficient to:

- use the new values of $A, \ldots,$ to compute the new "optimal" intervals $\mathbf{L}_{\text{new}}$ and $\mathbf{K}_{\text{new}}$; and

- check whether $L_{\mathrm{old}} \in \mathbf{L}_{\mathrm{new}}$ and $K_{\mathrm{old}} \in \mathbf{K}_{\mathrm{new}}$.

Equilibrium market: we need a new algorithm. For equilibrium market, the corresponding formulas (7′) and (8′) can overestimate. So:

- if $L_{\mathrm{old}} \notin \mathbf{L}_{\mathrm{new}}$ or $K_{\mathrm{old}} \notin \mathbf{K}_{\mathrm{new}}$, then we definitely need to change the production parameters; but
- if $L_{\mathrm{old}} \in \mathbf{L}_{\mathrm{new}}$ and $K_{\mathrm{old}} \in \mathbf{K}_{\mathrm{new}}$, it does not necessarily mean that the old parameters are still possibly optimal: it can be that they are no longer possibly optimal, but they nevertheless belong to the "overestimated" intervals (7′) and (8′).

To check whether a change is needed or not, we, therefore, need a new algorithm.

Equilibrium market: towards a new algorithm. We are given the intervals $\mathbf{A}, \alpha^*, \ldots, \mathbf{Q}$, and the (old) values K and L. We need to check whether these values are still possibly optimal, i.e., where there exist $A \in \mathbf{A}, \alpha \in \alpha^*, \ldots, Q \in \mathbf{Q}$ for which these K and L are optimal, or, equivalently, for which the equations (1) and (5) are both true. Let us describe these conditions (1) and (5) in terms of α and β.

If we apply logarithm to both sides of the equation (1), we conclude that

$$\ln(L) \cdot \alpha + \ln(K) \cdot \beta = \ln(Q) - \ln(A).$$

The coefficients at α and β in the left-hand side are known constants. Since $\ln(x)$ is strictly increasing, the right-hand side takes its smallest possible value when Q takes its smallest possible value and A takes the largest, i.e., when $Q = \underline{Q}$ and $A = \overline{A}$. Similarly, the right-hand side takes the *largest* possible value when $Q = \overline{Q}$ and $A = \underline{A}$. Thus, the existence of Q and A for which this equality is true is equivalent to the following double inequality:

$$\ln(\underline{Q}) - \ln(\overline{A}) \leq \ln(L) \cdot \alpha + \ln(K) \cdot \beta \leq \ln(\overline{Q}) - \ln(\underline{A}). \tag{11}$$

The equation (5) can be re-written as

$$\frac{a}{b} = \frac{\alpha}{\beta} \cdot \frac{K}{L}.$$

For given α and β, satisfying this equality means that its right-hand side must be within the interval of possible values of the ratio a/b, i.e. (since a and b are positive), within the interval $[\underline{a} / \overline{b}, \overline{a} / \underline{b}]$:

$$\underline{a} / \overline{b} \leq \frac{\alpha}{\beta} \cdot \frac{K}{L} \leq \overline{a} / \underline{b}.$$

If we multiply both sides of each inequality by β, we get two equivalent inequalities that are *linear* in α and β:

$$(\underline{a} / \overline{b}) \cdot \beta \leq \frac{K}{L} \cdot \alpha; \tag{12}$$

$$\frac{K}{L} \cdot \alpha \le (\overline{a} / \underline{b}) \cdot \beta. \tag{13}$$

Finally, the conditions that $\alpha \in \alpha^*$ and $\beta \in \beta^*$ can also be represented in terms of linear inequalities:

$$\underline{\alpha} \le \alpha \le \overline{\alpha}; \tag{14}$$

$$\underline{\beta} \le \beta \le \overline{\beta}. \tag{15}$$

Thus, K and L are possibly optimal if and only if the system of linear inequalities (11)–(15) has a solution. The existence of such a solution can be easily checked by linear programming methods. Hence, we arrive at the following method:

Equilibrium market: a new algorithm. *Given*: (old) values L and K, and new intervals $\mathbf{A}$, α^*, β^*, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{Q}$.

To check whether the values L and K are possibly optimal, we must apply linear programming to check whether a system of inequalities (11)–(15) is consistent (i.e., has a solution).

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