Global Optimization Using Interval Analysis: The One-Dimensional Case

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Abstract. We show how interval analysis can be used to compute the minimum value of a twice continuously differentiable function of one variable over a closed interval. When both the first and second derivatives of the function have a finite number of isolated zeros, our method never fails to find the global minimum.

Key Words. Global optimization, interval analysis, global minimization, one-dimensional optimization.

1. Introduction

Consider a function $f(x)$ in C^2. We shall describe a method for computing the minimum value of $f(x)$ on a closed interval $[a, b]$. We shall see that, if $f'(x)$ and $f''(x)$ have only a finite number of isolated zeros, our method always converges. In a subsequent paper, we shall show how the method can be extended to the case in which x is a vector of more than one variable. Moreover, it will be extended to the constrained case, and a modified method will remove the differentiability condition. The present paper serves to introduce the necessary ideas.

In practice, we can only compute minima in a bounded interval. Hence, it is no (additional) restriction to confine our attention to a closed interval. The term *global minimum* used herein refers to the fact that we find the smallest value of $f(x)$ throughout $[a, b]$. We shall not mistake a local minimum for the global one.

Indeed, our method will usually not find local minima, unless forced to do so. Its efficiency would then be degraded if it did. In our method, we iteratively delete subintervals of $[a, b]$ until the remaining set is sufficiently small. These subintervals consist of points at which either $f(x)$ is proved to exceed the minimum in value or else the derivative is proved to be nonzero.

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The tool which enables us to do this is interval analysis. It is not necessary for the reader to be intimately familiar with interval analysis to understand our paper, since we describe its relevant property in the next section. However, we assume that the reader is familiar with its rudiments. The interested reader should consult Moore's book (Ref. 1). For other papers which consider the problem of global optimization, see Refs. 2–8.

2. Interval Analysis

Let $g(x)$ be a rational function of x. On a computer, we can evaluate $g(x)$ for a given value of x by performing a specified sequence of arithmetic operations involving only addition, subtraction, multiplication, and division.

Let X be a closed interval. If we use X for input, instead of x, and perform the same sequence of operations using interval arithmetic (see Ref. 1), rather than ordinary real arithmetic, we obtain a closed interval $g(X)$ containing the range

$$\{g(x) : x \in X\}$$

of $g(x)$ over X. This result will not be sharp, in general; but, if outward rounding (see Ref. 1) is used, then $g(X)$ will always contain the range.

The lack of sharpness depends on many things; but, for exact interval arithmetic, it goes to zero as the width of X goes to zero.

If $g(x)$ is not rational, we assume that an algorithm is known for computing an interval $g(X)$ containing the range of $g(x)$ for $x \in X$. Methods for doing this are discussed in Ref. 1 and elsewhere.

3. Taylor's Theorem

We shall use interval analysis in conjunction with Taylor's theorem in two ways. First, we expand $f(x)$ as

$$f(y) = f(x) + (y - x)f'(x) + \tfrac{1}{2}(y - x)^2 f''(\zeta_1). \tag{1}$$

This holds for some number ζ_1 between x and y. If x and y are contained in an interval X, then ζ_1 is in X. Thus, if x is a given point in X, then

$$f(y) \in f(x) + (y - x)f'(x) + \tfrac{1}{2}(y - x)^2 f''(X), \tag{2}$$

for all $y \in X$.

We could relax our condition that $f(x) \in C^2$ and assume only that $f(x) \in C^1$. Instead of (2), we could then use a relation of the form

$$f(y) = f(x) + (y - x)f'(\zeta_2).$$

However, to increase the efficiency of our method, we assume that $f(x) \in C^2$ and use (2).

We shall, however, use an expansion of this form with f replaced by f'. We then have

$$f'(y) = f'(x) + (y - x)f''(\zeta), \tag{3}$$

$$f'(y) \in f'(x) + (y - x)f''(X). \tag{4}$$

This holds for all $y \in X$, if $x \in X$.

4. Approximate Value of the Global Minimum

As we proceed with our algorithm, we shall evaluate $f(x)$ at various points x in $[a, b]$. Let $\bar{f}$ denote the currently smallest value of f found so far.

We begin our algorithm by evaluating f at the endpoints of $[a, b]$. We initially choose

$$\bar{f} = \min[f(a), f(b)].$$

We shall record $\bar{f}$ and the point(s) x at which $f(x) = \bar{f}$.

Any minimum in the interior of $[a, b]$ must be a stationary point. Our algorithm will delete iteratively subintervals of $[a, b]$ wherein $f' \neq 0$. It will also delete subintervals wherein $f > \bar{f}$, since this implies that $f > f^*$, where f^* denotes the globally minimum value of f in $[a, b]$.

5. Concavity

As our algorithm proceeds, we shall dynamically subdivide $[a, b]$ into subintervals. Let X denote such a subinterval, and denote the interval resulting from evaluating f'' in interval arithmetic using the argument X by $[u, v]$; that is,

$$f''(X) = [u, v].$$

If $v < 0$, then f is concave in X and cannot have a minimum in the interior of X. Therefore, when our algorithm produces a subinterval X, we evaluate $f''(X)$. If $v < 0$, we delete X. Note that it is not necessary to examine the endpoints of such an interval. An endpoint cannot be a global minimum unless it is one of the points a or b already examined.

If $v \geq 0$, we use the information about f'' to eliminate all or part of X as described in the following sections.

6. Interval Newton Method Applied to f'

Let X be a closed subinterval of $[a, b]$. Let x be any point in X. It is best to let x be the midpoint of X. From (3), if $y \in X$ is a stationary point of f, then $f'(y) = 0$ and y solves the equation

$$f'(x) + (y - x)f''(\zeta) = 0, \tag{5}$$

for some value of $\zeta \in X$. Hence, any stationary point $y \in X$ is contained in the set

$$S'' = \{y : f'(x) + (y - x)f''(\zeta') = 0, \ \zeta' \in X\}$$

obtained by letting ζ' range over all values in X. We now make use of $f''(X)$, which we will have obtained while testing to see if f is concave in X (see Section 5). If $0 \notin f''(X)$, then the set S'' is contained in the set

$$S' = x - f'(x)/f''(X), \tag{6}$$

where the right-hand member of (6) is obtained by simple evaluation using interval arithmetic.

This is the interval Newton method as derived in Ref. 1. However, Moore uses it to find zeros of f, rather than zeros of f'. If we define the iterative process

$$N(X_n) = x_n - f'(x_n)/f''(X_n), \qquad n = 0, 1, \ldots,$$
$$X_{n+1} = X_n \cap N(X_n) \tag{7}$$

with x_n the midpoint of X_n, then the process never fails to converge (see Ref. 9) when $0 \notin f''(X_0)$, in the sense that, if there is a root x' of f' in X_0, then $X_n \to x'$ as $n \to \infty$. If f' has no root in X_0, then X_n is empty for a sufficiently large value of n. In the fomer case, the convergence is quadratic (see Ref. 11) in that, asymptotically,

$$w_{n+1} = O(w_n^2),$$

where w_n denotes the width of X_n.

It does not seem to have been previously observed, but the set S'' is easily bounded even when $0 \in f''(X)$. A solution in this case is derived in Ref. 10. The result is as follows.

Denote

$$X = [x_L, x_R] \qquad \text{and} \qquad f''(X) = [u, v].$$

If $u \neq 0$, define

$$c = x - f'(x)/u;$$

if $v \neq 0$, define

$$d = x - f'(x)/v.$$

We assume that $u \leq 0 \leq v$, since otherwise we can use the standard Newton formula (6). We are interested only in points of S'' which lie in X. Thus, we want the intersection of X with the set that we can obtain to bound S''. As shown in Ref. 10, these points compose two intervals. For $f'(x) \geq 0$, one is

$$S_1 = \begin{cases} [x_L, d], & \text{if } v > 0 \text{ and } d \geq x_L, \\ \Phi, & \text{if } v = 0 \text{ or if } v > 0 \text{ and } d < x_L, \end{cases} \tag{8-1}$$

(where Φ is the empty set) and the other is

$$S_2 = \begin{cases} [c, x_R], & \text{if } u < 0 \text{ and } c \leq x_R, \\ \Phi, & \text{if } u = 0 \text{ or if } u < 0 \text{ and } c > x_R. \end{cases} \tag{8-2}$$

For $f'(x) \leq 0$ these intervals are

$$S_1 = \begin{cases} [x_L, c], & \text{if } u < 0 \text{ and } c \geq x_L, \\ \Phi, & \text{if } u = 0 \text{ or if } u < 0 \text{ and } c < x_L, \end{cases} \tag{8-3}$$

and

$$S_2 = \begin{cases} [d, x_R], & \text{if } v > 0 \text{ and } d \leq x_R, \\ \Phi, & \text{if } v = 0 \text{ or if } v > 0 \text{ and } d > x_R. \end{cases} \tag{8-4}$$

The solution set of interest is $S = S_1 \cup S_2$, which may be empty, or consist of a single interval, or consist of two disjoint intervals. As shown in Ref. 10,

$$X \cap S'' \subset S.$$

Whether $0 \in f''(X)$ or not, we now have a method for obtaining a set S containing any zero of f' in X. Hence, its complement S^c in X cannot contain a minimum of f, and we can discard S^c. Note that, if S is empty, we discard all of X. We could now repeat the Newton step with S in place of X. If S consists of two intervals, we would use each one separately. Thus, we would evaluate f'' with argument S if S is a single interval, or with argument S_1 (or S_2) otherwise. We would then use either (6) or (8), whichever is appropriate, and iterate.

In Ref. 10, the following theorem is proved which expresses the convergence of this extended Newton method.

Theorem 6.1. Let the extended Newton method be applied to finding the zeros of $f'(x)$ in an interval X_0. Assume that, at the ith step, $i = 1, 2, 3, \ldots$, the method is applied to the largest remaining subinterval. Also

assume that $f'(x)$ is continuous and has a finite number of distinct zeros in X_0. Then, for a sufficiently large value of i, the sum of the lengths of the remaining intervals is less than an arbitrarily prescribed number $\epsilon > 0$.

We now sketch the outline of a proof of this theorem. For a complete but rather lengthy proof, see Ref. 10.

We ignore the trivial case in which x_n is a zero of $f'(x)$. It then follows than x_n is not contained in $N(X_n)$. For, if it were, there would be a point $y \in X_n$ such that [see Eq. (7)]

$$x_n = x_n - f(x_n)/f'(y).$$

This implies that $f(x_n) = 0$, which is a contradiction. When $0 \notin f'(X_n)$, this implies that the width of $N(X_n)$ is less than half that of X_n. When $0 \in f'(X_n)$, the Newton step given by (7) yields two subintervals of width less than half that of X_n. In subsequent steps, these subintervals are again reduced in size. Eventually, all remaining subintervals become so small that none contains more than one zero of $f'(x)$. It is proven in Ref. 10 that any interval not containing a zero of $f'(x)$ is rapidly eliminated by the interval Newton process. The width of the remaining intervals goes to zero asymptotically.

In practice, however, it will generally be more efficient not to simply continue using the extended interval Newton method. For, it would isolate all the stationary points of $f(x)$ in $[a, b]$. Hence, we would expend effort in finding a local (nonglobal) minimum, in which we are not interested. Instead, we supplement the Newton method using the following procedure.

7. Bounding f

We now consider how to use (1) to delete points $y \in X$ where $f(y) > \bar{f}$, and hence where $f(y)$ is not a global minimum. For simplicity, we shall discuss the procedure as if we delete points where $f(y) \geq \bar{f}$ (i.e., where equality can hold). However, we shall retain closed intervals, and thus points where $f(y) = \bar{f}$ will not be discarded.

To speed convergence, we can allow a prescribed error $\epsilon_1 \geq 0$ and delete points for which

$$f(y) \geq \bar{f} - \epsilon_1. \tag{9}$$

Our discussion will be valid for ϵ_1 either zero or nonzero. We can allow ϵ_1 to be nonzero only if we do not need to know the point(s) x^* at which f is globally minimum. We discuss this aspect later.

From (1), condition (9) holds if

$$f(x) + (y - x)f'(x) + \tfrac{1}{2}(y - x)^2 f''(\zeta) \geq \bar{f} - \epsilon_1.$$

We choose x to be the center of the interval X. Denote

$$E = \bar{f} - f(x) - \epsilon_1 \tag{10}$$

and, as before, denote

$$f''(X) = [u, v].$$

We want to discard points y such that

$$(y - x)f'(x) + \tfrac{1}{2}(y - x)^2 z \geq E \tag{11}$$

for all $z \in [u, v]$. The left-hand member increases as z increases. Hence, if (11) is satisfied for $z = u$, it is satisfied for all $z \in [u, v]$. Therefore, we need only calculate those values of y satisfying

$$(y - x)f'(x) + \tfrac{1}{2}(y - x)^2 u \geq E. \tag{12}$$

Consider the discriminant

$$\Delta = [f'(x)]^2 + 2Eu.$$

If $\Delta < 0$, the quadratic has no real roots. Hence, since (12) is satisfied for $y = x$, it is satisfied for all y. Thus, we can eliminate all of X.

If $u = 0$, (12) is satisfied for

$$y \geq x + E/f'(x) \qquad \text{if } f'(x) > 0,$$

$$y \text{ arbitrary if } f'(x) = 0,$$

$$y \leq x + E/f'(x) \qquad \text{if } f'(x) < 0.$$

If $u \neq 0$ and $\Delta > 0$, the quadratic in (12) has two real roots, say r_1 and r_2. Thus, (12) is satisfied for y between r_1 and r_2 if $u < 0$, and for y outside the interval defined by r_1 and r_2 if $u > 0$.

We have determined the values of y for which

$$f(y) \geq \bar{f} - \epsilon_1.$$

The set of these values of y which lie in X can be discarded. It is necessary to confine these points to X, since our information about f'' is for points in X only.

8. Updating $\bar{f}$

The process described in Section 7 involves the evaluation of $f(x)$, where x is the center of X. We use this information to update $\bar{f}$. If $f(x) < \bar{f}$, we replace $\bar{f}$ by $f(x)$. Thus, $\bar{f}$ is always the smallest value of f currently known.

9. Choice of ϵ_1

If we are only interested in finding the value of f^*, but not x^*, we can choose $\epsilon_1 > 0$. But, if we want to determine x^*, we must choose $\epsilon_1 = 0$.

To see this, suppose that

$$\bar{f} - f^* < \epsilon_1.$$

Using the procedure in Section 7 with $\epsilon_1 > 0$ may eliminate a set of points y containing x^*, but we have no way of knowing this. If we choose $\varepsilon_1 = 0$, we will never delete x^*. A point x^* will eventually be isolated when the set of points remaining (not yet deleted) is small. For $\epsilon_1 > 0$, we can iterate until the entire interval $[a, b]$ is deleted. Only the point(s) $\bar{x}$ at which $f(\bar{x}) = \bar{f}$ will be retained. At these points, we shall have $\bar{f} - f^* < \epsilon_1$. That is, we shall without fail determine f^* to within the prescribed error ϵ_1.

A compromise is possible which may improve efficiency. We can use $\epsilon_1 > 0$ anticipating that $\bar{f}$ will become smaller as we proceed.

Thus, if we currently have a value $\bar{f}_0$ for $\bar{f}$, but anticipate that $\bar{f}$ will eventually be reduced to $\bar{f}_1$, we could choose ϵ_1 somewhat less than $\bar{f}_0 - \bar{f}_1$. Then, if $\bar{f}$ does become as small as $\bar{f}_1$, we will not have deleted a point where f is globally minimum. However, if $\bar{f}$ remains greater than $\bar{f}_1$, we will have to repeat whose steps which used too large a value of ϵ_1. We have not tried this procedure in practice.

10. List of Intervals

When we begin our algorithm, we will have a single interval $[a, b]$. In our first step, we use this entire interval as the interval X discussed in previous sections. As shown in Ref. 10, as long as u and v are bounded, our procedure will eliminate a part of this interval which, initially, we can expect to be a small subinterval, say Y, of $[a, b]$.

The exterior of Y in X consists (in this case) of two subintervals of X. They constitute an initial list of intervals yet to be processed. When we process one of these intervals, we may get two more. If so, we add them to the list. Thus, the number of intervals in the list tends to grow initially.

Eventually, however, the intervals become small enough that either only one new interval is obtained from an interval X or else X is entirely eliminated. Thus, the number of intervals in the list eventually decreases to one or a few intervals, depending on the number of points at which f takes on its globally minimum value. We assume for simplicity that there are a small number of such points.

When we choose a new interval from the list to process by our method, it is best to choose the largest one in the list. This is because we want to obtain quickly a value of $\bar{f}$ near f^*. The smaller $\bar{f}$ is, the greater portion of an interval X we shall be able to delete using (12).

If an interval X in the list is small, we can expect to have previously evaluated f at a point near X. Hence, we cannot expect $\bar{f}$ to decrease very much when we evaluate f in X.

If the function f (and/or its derivatives) is expensive in time to evaluate, it will pay to search the list and find the largest interval. But, if f and its derivatives require little effort to evaluate, the search procedure itself may be too time-consuming. A compromise that is easy to program is to use the *oldest* interval in the list.

One way to get $\bar{f}$ small initially and obviate the need to worry about the size of the intervals is to begin the entire process with a noninterval method for finding a local minimum. This gives a small value of $\bar{f}$ initially.

If our interval method subsequently finds a better value of $\bar{f}$, the local method could then be used to search in the neighborhood of the point in question.

These variations of the method are not essential. They decrease the running time of our algorithm by at most a few percent in practice.

11. Options in the Method

In Sections 5 and 6, we showed how to compute a set S^c which could be deleted because

$$f'(y) \neq 0, \quad \text{for } y \in S^c.$$

In Section 7, we showed how to compute a set, say S', which could be deleted because

$$f(y) > \bar{f} - \epsilon_1, \quad \text{for } y \in S'.$$

These sets may have considerable overlap. In fact, it is not unusual for one of the sets to be contained in the other.

Hence, one may ask whether it is desirable to always use both methods. It is certainly possible to find the global minimum using either method alone. The best procedure might be to use only one of the methods at a given step and somehow predict which will delete the largest set.

However, if the function f and/or its derivatives are costly to evaluate, the additional work to apply the two methods will be relatively small. We have taken the easy way out and made this assumption. Thus, we use both methods at each step.

12. Termination

We now consider methods for terminating our algorithm. In Section 7, we pointed out that, if we were not interested in the point(s) where f was minimum, but we wanted only to approximate the global minimum f^*, then we could choose $\epsilon_1 > 0$. We can then continue the iteration until all of the initial interval is deleted. The final value of $\bar{f}$ exceeds f^* by no more than ϵ_1.

If we choose $\epsilon_1 > 0$, we cannot determine where f is minimal. If we do want to know where f is minimal, we can iterate (with $\epsilon_1 = 0$) until the remaining set of intervals in the list have combined length less than, say, ϵ_2. Thus, if at some stage there are s intervals

$$X_i = [a_i, b_i], \qquad i = 1, \ldots, s,$$

in the list, we require that

$$\sum_{i=1}^{s} (b_i - a_i) \leq \epsilon_2. \tag{13}$$

This presupposes that the set of points at which f is minimal is finite. We assume that this is the case. If it is not the case, we can iterate until

$$b_i - a_i \leq \epsilon_2$$

for each $i = 1, \ldots, s$.

We may wish assurance that none of the final intervals X_i is such that $f(x)$ is considerably larger than f^* for all x in X_i. We can obtain this assurance by evaluating f at the midpoint x_i of each X_i. If $f(x_i)$ differs from $\bar{f}$ by too large an amount, we can process the interval X_i by our method until it is either entirely deleted or else a value of f sufficiently close to $\bar{f}$ is found.

We consider three types of minimization problems: (i) bound x^* with error tolerance ϵ_2; (ii) bound f^* with error tolerance ϵ_1; and (iii) bound both f^* with error tolerance ϵ_1 and x^* with error tolerance ϵ_2. Our method solves the first problem if we use the termination criterion (13) discussed above.

It solves the second problem if we choose $\epsilon_1 > 0$ in (10) and iterate until the initial interval is entirely eliminated. The final value of $\bar{f}$ then satisfies

$$\bar{f} - f^* \leq \epsilon_1.$$

The third problem requires some additional computation. We first proceed as in the first method until x^* is bounded as required. We then bound f in each remaining interval by the method described in the next section. If the bound does not satisfy the error tolerance in one of these intervals, we process the interval again by our algorithm described in preceding sections. We continue until the error criterion on f^* is satisfied for all remaining intervals.

If we could use exact interval arithmetic, the choice of ϵ_1 and ϵ_2 would be arbitrary. The theorem in Section 6 assures that our method would always reach the stage at which the termination criterion or criteria was satisfied. In practice, it is necessary to choose ϵ_1 and/or ϵ_2 commensurate with the attainable accuracy of the computer.

13. Bounding f^*

We now consider how to bound f, and hence f^*, in the intervals remaining after (13) is satisfied. Let X_i be such an interval, and let x_i denote its center. If we replace x by x_i and y by X_i in the right-hand member of (2), we have

$$f(y) \in f(x_i) + (X_i - x_i)f'(x_i) + \tfrac{1}{2}(X_i - x_i)^2 f''(X_i). \tag{14}$$

We evaluate the right-hand member of (14) using interval arithmetic and obtain an interval, say, $[p_i, q_i]$. Then,

$$p_i \le f(y) \le q_i$$

for all $y \in X_i$.

Let s denote the number of intervals X_i remaining, and denote

$$p = \min_{1 \le i \le s} p_i.$$

Then,

$$p \le f^* \le \bar{f}. \tag{15}$$

If $\bar{f} - p \le \epsilon_1$, we are finished. If $\bar{f} - p > \epsilon_1$, let j denote the index of an interval for which $p = p_j$. We process X_j by our algorithm and continue until $\bar{f} - p \le \epsilon_1$. Then, from (15), our error criterion is satisfied, and we have finished.

14. Rounding Errors

In practice, we shall make rounding errors in applying our method. This poses no problem in obtaining such quantities as $f''(X)$. We simply follow the standard practice in doing interval arithmetic and use outward rounding (see Ref. 1). Thus, for example, if we are computing $f''(X)$, the actual interval $[u, v]$ obtained contains $f''(X)$. Consequently, the steps taken in our method which use $[u, v]$ yield correct results. The effect of the rounding errors is merely to slow convergence very slightly.

To assure that rounding errors do not produce incorrect results in our method, it is necessary to compute quantities such as $f(x)$ or $f'(x)$ in interval arithmetic. Again, this creates no difficulty when such quantities are used in evaluation of a formula, such as the one for the Newton step, which already involves interval quantities.

In Section 8 and elsewhere, we discussed the quantity $\bar{f}$ as if it were calculated exactly. What we want for $\bar{f}$ is a quantity such that, without fail, $f^* \le \bar{f}$. This is because we delete points x where $f(x) \ge \bar{f}$, and we do not want to delete more points than we should. In practice, we evaluate $f(x)$ in interval arithmetic and obtain, say, $[f_L(x), f_R(x)]$. Therefore, when we evaluate $f(x)$, we update $\bar{f}$ by replacing $\bar{f}$ by $f_R(x)$ if $f_R(x) < \bar{f}$. Thus, we know that $f^* \le \bar{f}$.

15. Numerical Example

We now consider an example discussed by Schubert in Ref. 11. We shall find the global minimum of

$$f(x) = -\sum_{k=1}^{5} k \sin[(k+1)x + k].$$

We have changed the sign of his function, since we seek the global minimum, and he sought the global maximum.

Schubert points out that f has global minima at $x_1^* = -6.7745\ldots$ and $x_2^* = 5.7918\ldots$. This is correct. However, he states that f has a local (nonglobal) minimum at $x_3^* = -0.4914\ldots$. This is incorrect, as can be seen by noting that f has period 2π. The points x_1^* and x_3^* differ by 2π as do x_2^* and x_3^*. Hence, f has the same value at each of the three points. That is, x_3^* is also a point of global minimum.

He chooses the initial interval $[-10, 10]$ and notes that there are 18 minima of $f(x)$ in this interval, besides x_1^*, x_2^*, x_3^*. Since $f(x)$ has period 2π, we need only search in an interval of length 2π. However, for easier comparison, we shall also use $[-10, 10]$.

Actually, it is somewhat difficult to compare our method with Schubert's method. His method requires values of $f(x)$ only, while our method requires $f'(x)$ and $f''(X)$ also. However, he requires knowledge of a Lipschitz constant, which is not generally obtainable. A comparable part of our method is computation of the derivative of f, which obviates the need of a Lipschitz constant. Removing the need for this constant makes our method applicable to a wider variety of functions.

We shall simply compare the number of iterations. Even this comparison is not straightforward. His termination criterion is that f^* be

determined to within $\epsilon_1 = 0.01$. His method produces error bounds on the x^*, but their values have nothing to do with the termination criterion. Our situation is essentially the reverse of this. We use bounds on the x^* as our primary termination criterion, when we want bounds on both x^* and f^*.

His final error bounds on the three global minima sum to about 0.15. We chose the slightly more stringent condition $\epsilon_2 = 0.1$, along with the condition $\epsilon_1 = 0.01$. We obtained a much sharper result than required. To ten decimals, our final intervals were

$$x_1^* \in [-6.774576144, -6.774576143],$$

$$x_2^* \in [5.791789015, 5.791799064],$$

$$x_3^* \in [-0.4913921876, -0.4913895811].$$

These intervals have a combined length of approximately 1.3×10^{-5}.

These results required an evaluation of $f(x)$ at each endpoint of the original interval and 77 evaluations of $f(x), f'(x), f''(X)$. An additional three evaluations of these quantities used in (14) gave

$$f(x_1^*) \in [-12.03124944, -12.03124943],$$

$$f(x_2^*) \in [-12.03124945, -12.03124943],$$

$$f(x_3^*) \in [-12.3124944, -12.3124943].$$

Schubert's method required 444 evaluations of f.

These computations were done on the HP-9830 computer, which used twelve decimal digit calculations.

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