JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 58 (1995) 183–192

Computable error bounds for an optimization problem with parallelepiped constraint

Ismail Bin Mohd

Department of Mathematics, University Pertanian Malaysia, Serdang, 43400 UPM, Selangor Darul Ehsan, Malaysia

Received 26 March 1993; revised 24 August 1993

Abstract

It can be shown that interval arithmetic can be used to locate the Kuhn–Tucker point of a nonlinear programming problem where its constraint is a parallelepiped with sides parallel to the coordinate axes.

Keywords: Interval arithmetic; Kuhn–Tucker point; Parallelepiped constraint; Transformation

1. Introduction

In [9], Robinson has introduced a method for obtaining computable bounds of error in an approximate Kuhn–Tucker point [2] $z^* = (x^{*\mathrm{T}}, v^{*\mathrm{T}}, w^{*\mathrm{T}})^{\mathrm{T}}$ of a nonlinear programming problem of the form

$$\begin{aligned} &\text{minimize} \quad \Phi(x) \ (x \in \hat{D} \subseteq \mathbb{R}^n) \\ &\text{subject to} \quad c_i(x) \leqslant 0 \ (i = 1, \ldots, m), \\ &\qquad\qquad h_j(x) = 0 \ (j = 1, \ldots, r), \end{aligned} \tag{1.1}$$

where $\Phi : D \subseteq \mathbb{R}^n \to \mathbb{R}^1$, $c_i : \mathbb{R}^n \to \mathbb{R}^1$ $(i = 1, \ldots, m)$, $h_j : \mathbb{R}^n \to \mathbb{R}^1$ $(j = 1, \ldots, r)$ are given continuously differentiable functions and $\hat{D} \subset D$ is an open set containing the points which satisfy the constraints. In Robinson's method one chooses any initial interval vector z which might contain z^* and tries the method but if it does not work, adjusts z and tries again.

We prefer the nonlinear programming problem of the special form

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in D \subseteq \mathbb{R}^n, \end{aligned} \tag{1.2}$$

SSDI 0377-0427(93)E0201-V

where D is a parallelepiped with sides parallel to the coordinate axes in a manner similar to that which has been used by Robinson [9]. In contrast, without choosing any interval vector $\boldsymbol{z}$, we can show how to compute an initial interval vector of Lagrange multipliers easily using interval arithmetic [1, 7] which contains z^*.

In order to show how to compute an initial interval vector of Lagrange multipliers, we have arranged this paper as follows.

In Section 2, some definitions of interval arithmetic are given. The detailed discussion of interval arithmetic and its extensions can be seen in [1, 7].

In Section 3, the preliminary results of Kuhn–Tucker points are given, especially the theory concerning first-order constraint qualification. Section 4 contains the problem to be considered in this paper.

Since the minimizers of a constrained problem of the form (1.1) as mentioned in [2] correspond to their Kuhn–Tucker points, we only need to discuss that first-order constraint qualification holds at the minimizer(s) of the problem given in Section 4. The discussion is given in Section 5.

We describe in detail how to obtain the bounds of the Kuhn–Tucker point as well as the minimizer in Section 6 and give a brief application of the method to the example given in [4] (in Section 7) before concluding in Section 8.

2. Interval analysis

Definition 1. An interval $\boldsymbol{x} \in I(\mathbb{R}) = \{\text{intervals on real line}\}$ is denoted by $[x_{\text{I}}, x_{\text{S}}]$, where x_{I} and x_{S} are called infimum and supremum, respectively.

Definition 2. The binary arithmetic operations $+$, $-$, $\cdot$, and $/$ are defined on $I(\mathbb{R})$ according to

$$\boldsymbol{x} * \boldsymbol{y} = \{x * y \mid x \in \boldsymbol{x},\ y \in \boldsymbol{y};\ \boldsymbol{x}, \boldsymbol{y} \in I(\mathbb{R})\}$$

in which the asterisk belongs to the set $\{+, -, \cdot, /\}$ save that $\boldsymbol{x}/\boldsymbol{y}$ is not defined if $0 \in \boldsymbol{y}$.

Definition 3. An $n \times 1$ interval vector which is called *box* $\boldsymbol{x} = (\boldsymbol{x}_i)_{n \times 1} \in I(\mathbb{R}^n)$ has ith element $\boldsymbol{x}_i = [x_{i\text{I}}, x_{i\text{S}}] \in I(\mathbb{R})$.

3. Preliminary results

Consider the nonlinear programming problems of the form

$$\begin{aligned} &\text{minimize} \quad f(x) \quad (x \in \hat{D} \subseteq \mathbb{R}^n) \\ &\text{subject to} \quad c_i(x) \geqslant 0 \quad (i = 1, \ldots, m), \end{aligned} \tag{3.1}$$

where $f: D \subseteq \mathbb{R}^n \to \mathbb{R}^1$, $c_i: \mathbb{R}^n \to \mathbb{R}^1$ $(i = 1, \ldots, m)$ are given continuously differentiable functions and $\hat{D} \subset D$ is an open set containing the points which satisfy the constraints.

Definition 4. The Lagrangean function [2] $L:\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^1$ corresponding to (3.1) is defined by

$$L(x,u) = f(x) - \sum_{i=1}^{m} u_i c_i(x). \tag{3.2}$$

Suppose that $x^* \in \mathbb{R}^n$ is a feasible point for (3.1) and that $f:\mathbb{R}^n \to \mathbb{R}^1$, and $c_i:\mathbb{R}^n \to \mathbb{R}^1$ $(i = 1, \dots, m)$ have first partial derivatives at x^*. Let $c_i^* = c_1(x^*)$ $(i = 1, \dots, m)$, etc. and let

$$B^* = \{i \in \mathbb{N}^+ \mid c_i(x^*) = 0\}, \tag{3.3}$$

$$Z_1^* = \{z \in \mathbb{R}^n \mid z^{\mathrm{T}}\nabla c_i^* \geqslant 0 \ (\forall i \in B^*) \wedge z^{\mathrm{T}}\nabla f^* \geqslant 0\}, \tag{3.4}$$

$$Z_2^* = \{z \in \mathbb{R}^n \mid z^{\mathrm{T}}\nabla c_i^* \geqslant 0 \ (\forall i \in B^*) \wedge z^{\mathrm{T}}\nabla f^* < 0\}, \tag{3.5}$$

$$Z_3^* = \{z \in \mathbb{R}^n \mid \exists\, i \in B^*,\ z^{\mathrm{T}}\nabla c_i^* < 0\}, \tag{3.6}$$

where $\nabla c_i^* = \nabla c_i(x^*)$ $(i = 1, \dots, m)$ and $\nabla f^* = \nabla f(x^*)$. The sets Z_1^*, Z_2^* and Z_3^* are disjoint and $Z_1^* \cup Z_2^* \cup Z_3^* = \mathbb{R}^n$.

Observe that all feasible directions from x^* must be contained in $Z_1^* \cup Z_2^*$. Furthermore, $f(x)$ initially decreases along $z \in Z_2^*$ and initially increases or is constant along $z \in Z_1^*$. Thus if $Z_2^* \neq \emptyset$ we would not expect x^* to be a local minimizer [2].

The following theorem states that Lagrange multipliers exist generally [2].

Theorem 5. *If* (1) $x^* \in \mathbb{R}^n$ *is a feasible point for* (3.1); (2) $f \in C^1(\hat{D})$ *and* $c_i \in C^1(\hat{D})$ $(i = 1, \dots, m)$ *where* $\hat{D} \subset D$ *is an open set containing the points which satisfy the constraints*; (3) $Z_2^* = \emptyset$, *then* $\exists\ u^* \in \mathbb{R}^m$ *such that*

$$c_i(x^*) \geqslant 0 \quad (i = 1, \dots, m), \tag{3.7}$$

$$u_i^* c_i(x^*) = 0 \quad (i = 1, \dots, m), \tag{3.8}$$

$$u_i^* \geqslant 0 \quad (i = 1, \dots, m), \tag{3.9}$$

$$\nabla_x L(x^*, u^*) = 0. \tag{3.10}$$

Definition 6. The point $(x^{*\mathrm{T}}, u^{*\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{n+m}$ is a *Kuhn–Tucker* (KT) *point* for (3.1) if and only if (3.7)–(3.10) hold.

In applying Theorem 5 one must be able to determine whether the set Z_2^* is empty or not.

Definition 7. Let $x^* \in \mathbb{R}^n$ be a feasible point for (3.1) and suppose that $c_i \in C^1(\hat{D})$ $(i = 1, \dots, m)$, where $\hat{D} \subset D$ is an open set containing the points which satisfy the constraints. The *first-order constraint qualification* holds at x^* if and only if

$$((z \neq 0) \wedge (z^{\mathrm{T}}\nabla c_i^* \geqslant 0 \ (\forall i \in B^*)))$$

implies that z is tangential to a once differentiable arc emanating from x^* and contained in the feasible region.

The first-order constraint qualification is one of the best-known conditions implying that $Z_2^* = \emptyset$ as mentioned in the following theorem whose proof can be seen in [2].

Theorem 8. *If* (1) $x^* \in \mathbb{R}^n$ *is a feasible point for* (3.1); (2) $f \in C^1(\hat{D})$ *and* $c_i \in C^1(\hat{D})$ $(i = 1, \ldots, m)$, *where* $\hat{D} \subset D$ *is an open set containing the points which satisfy the constraints;* (3) *the first-order constraint qualification holds at* x^*, *then* $\exists\, u^* \in \mathbb{R}^m$ *such that* $(x^{*\mathrm{T}}, u^{*\mathrm{T}})^{\mathrm{T}}$ *is a Kuhn–Tucker* (*KT*) *point for* (3.1).

4. The nonlinear programming problem expressed as a system of nonlinear equations and inequalities

Consider the following special case of the problem (3.1):

$$\begin{aligned} &\text{minimize} \quad f(x) \ (x \in \hat{D} \subseteq D \subseteq \mathbb{R}^n) \\ &\text{subject to} \quad c_i(x) \geqslant 0 \ (i = 1, \ldots, m), \end{aligned} \tag{4.1}$$

in which

$$c_i(x) = x_i - \hat{x}_{i1} \quad (i = 1, \ldots, n), \tag{4.2}$$

$$c_i(x) = \hat{x}_{i-n\mathrm{S}} - x_{i-n} \quad (i = n + 1, \ldots 2n), \tag{4.3}$$

where $\hat{x} \in I(\hat{D})$ is given, so that $m = 2n$. The problem of bounding the solutions of (4.1) is equivalent to problem (1.2).

5. First-order constraint qualification

In this section, it will be shown that the first-order constraint qualification holds at the minimizer of (4.1) which is a Kuhn–Tucker point as discussed in Section 3.

In order to obtain the computable bounds of Lagrange multipliers for the minimizer of (4.1) we need to discuss its strict complementary slackness as given at the end of this section.

We need the following theorem whose proof can be seen in [2].

Theorem 9. *If* (1) $x^* \in \mathbb{R}^n$ *is a feasible solution of* (4.1); (2) $f \in C^1(\hat{D})$, $c_i \in C^1(\hat{D})$ $(i = 1, \ldots, 2n)$, *where* $\hat{D} \subset D$ *is an open set containing the points which satisfy the constraints;* (3) $\{\nabla c_i^* \mid i \in B^*\}$ *is linearly independent, then the first-order constraint qualification holds at* x^*.

The solution of (4.1) lies either in the interior of $\hat{x}$ (int($\hat{x}$)) or on the boundary of $\hat{x}$ ($\beta(\hat{x})$). For example suppose that $n = 2$.

If $x^* \in \mathrm{int}(\hat{x})$ then (see (3.3)) $B^* = \emptyset$ since by using (4.2) and (4.3), $c_i(x^*) > 0$ $(i = 1, \ldots, 4)$. Therefore, according to Definition 7 as explained in [5], the first-order constraint qualification holds at $x^* \in \mathrm{int}(\hat{x})$.

If $x^* \in \beta(\hat{x})$ then by (4.2) and (4.3), we have

$$B^* \in \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}\}. \tag{5.1}$$

Therefore, according to Theorem 9 as explained in [5], the first-order constraint qualification holds at $x^* \in \beta(\hat{x})$. A similar argument is valid for $n > 2$.

We conclude that the first-order constraint qualification holds at x^* for (4.1). Therefore, by Theorems 8, 5 and Definition 6, $\exists u^* \in \mathbb{R}^{2n}$, such that

$$F(z^*) = 0, \tag{5.2}$$

$$c(x^*) \geqslant 0, \tag{5.3}$$

$$u^* \geqslant 0, \tag{5.4}$$

where $z^* = (x^{*\mathrm{T}}, u^{*\mathrm{T}})^{\mathrm{T}}$, and $F: \mathbb{R}^{3n} \to \mathbb{R}^{3n}$, $c: \mathbb{R}^n \to \mathbb{R}^{2n}$ are defined by

$$F(z) = \begin{bmatrix} (f'(x) - u^{\mathrm{T}}c'(x))^{\mathrm{T}} \\ u_1 c_1(x) \\ \vdots \\ u_{2n} c_{2n}(x) \end{bmatrix}, \tag{5.5}$$

$$c(x) = (c_1(x), \ldots, c_{2n}(x))^{\mathrm{T}}. \tag{5.6}$$

The Kuhn–Tucker points for (4.1) which correspond to the local minimizers of (1.2) satisfy

$$F(z) = 0, \tag{5.7}$$

$$c(x) \geqslant 0, \tag{5.8}$$

$$u \geqslant 0. \tag{5.9}$$

Definition 10. *Strict complementary slackness* is said to hold at a Kuhn–Tucker point $z^* = (x^{*\mathrm{T}}, u^{*\mathrm{T}})^{\mathrm{T}}$ if and only if for $i = 1, \ldots, 2n$, $u_i^* > 0$ if $c_i(x^*) = 0$ and $u_i^* = 0$ if $c_i(x^*) > 0$.

Definition 10 is used as follows.

If $x^* \in \mathrm{int}(\hat{x})$ then by (4.2) and (4.3) $c_i(x)^* > 0$ $(i = 1, \ldots, 2n)$. Therefore, according to Definition 10, we have $u_i^* = 0$ $(i = 1, \ldots, 2n)$.

If $x_i^* = \hat{x}_{iI}$ $(i \in \{1, \ldots, n\})$ then by (4.2) and (4.3) $c_i(x^*) = 0$ and $c_{i+n}(x^*) > 0$, respectively. Therefore according to Definition 10, we have $u_i^* > 0$ and $u_{i+n}^* = 0$.

The detailed explanation of strict complementary slackness is given in [5].

Theorem 11. (a) *If* $x^* \in \mathrm{int}(\hat{x})$ *is a solution of* (4.1) *then strict complementary slackness holds at* x^*; (b) *if* $x^* \in \beta(\hat{x})$ *is a solution of* (4.1) *then strict complementary slackness holds at* x^* *if and only if* $\partial_i f(x^*) > 0$ $\forall i \in \{1, \ldots, n\}$ *such that* $x_i^* = \hat{x}_{iI}$ *and* $\partial_i f(x^*) < 0$ $\forall i \in \{1, \ldots, n\}$ *such that* $x_i^* = \hat{x}_{iS}$.

Proof. The proof is given in [5]. □

6. Bounding the Lagrange multipliers

Suppose that $\hat{z} = (\hat{x}^T, \hat{u}^T)^T$ is a box which is assumed to contain a Kuhn–Tucker point $z^* = (x^{*T}, u^{*T})^T$ for (4.1) where

$$\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^T \quad \text{and} \quad \hat{u} = (\hat{u}_i, \ldots, \hat{u}_{2n})^T.$$

We are given $\hat{x}$ but not $\hat{u}$. However, by using the interval extension of the derivative of the objective function in (4.1) we can determine $\hat{u}_i, \ldots, \hat{u}_{2n}$ which contain the corresponding Lagrange multipliers as shown in this section.

For (4.1) we have two cases, namely (i) $x^* \in \text{int}(\hat{x})$ and (ii) $x^* \in \beta(\hat{x})$.

In (i) $u_i = 0$ $(i = 1, \ldots, 2n)$ because no constraint is active at x^*. Therefore, we need to solve only

$$F(x) = 0, \tag{6.1}$$

where

$$F(x) = (\partial_i f(x))_{n \times 1}. \tag{6.2}$$

Therefore, we have

$$\hat{z} = (\hat{x}^T, \hat{0}^T)^T. \tag{6.3}$$

In (ii) $x^* \in \beta(\hat{x})$, so some of the Lagrange multipliers might be nonzero.

Now we derive an algorithm for obtaining the initial bounding intervals for the Lagrange multipliers.

Consider the first n equations of (5.5), namely

$$\partial_i f(z_1, \ldots, z_n) - z_{n+i} + z_{2n+i} = 0 \quad (i = 1, \ldots, n), \tag{6.4}$$

which is equivalent to

$$u_i - u_{i+n} = \partial_i f(x) \quad (i = 1, \ldots, n). \tag{6.5}$$

If $x \in \hat{x}$ then by (6.5)

$$u_i - u_{i+n} \in \partial_i \boldsymbol{f}(\hat{x}) \quad (i = 1, \ldots, n). \tag{6.6}$$

So if $\hat{d}_i = \partial_i \boldsymbol{f}(\hat{x})$ $(i = 1, \ldots, n)$ then by (6.6)

$$\hat{d}_{iI} \leqslant u_i - u_{i+n} \leqslant \hat{d}_{iS} \quad (i = 1, \ldots, n). \tag{6.7}$$

Furthermore, we know (see Section 5) that for $i = 1, \ldots, n$

$$(u_i > 0) \;\Rightarrow\; (u_{i+n} = 0), \tag{6.8}$$

that

$$(u_{i+n} > 0) \;\Rightarrow\; (u_i = 0), \tag{6.9}$$

and that

$$u_i \geqslant 0 \quad (i = 1, \ldots, 2n). \tag{6.10}$$

Suppose that

$$\hat{d}_{iS} \leqslant 0. \tag{6.11}$$

Then by (6.7),

$$u_i \leqslant u_{i+n}. \tag{6.12}$$

Suppose that $u_i > 0$. Then by (6.12) $u_{i+n} > 0$, contradicting (6.8). Therefore, if (6.11) holds then by (6.10) $u_i = 0$. Therefore, by (6.7) if (6.11) holds then

$$\hat{d}_{iI} \leqslant -u_{i+n} \leqslant \hat{d}_{iS} \leqslant 0,$$

whence by (6.10),

$$u_{i+n} \in [-\hat{d}_{iS}, -\hat{d}_{iI}] = -\hat{\boldsymbol{d}}_i.$$

Therefore,

$$(\hat{d}_{iS} \leqslant 0) \Rightarrow (u_i = 0 \wedge u_{i+n} \in -\hat{\boldsymbol{d}}_i). \tag{6.13}$$

Suppose that

$$\hat{d}_{iI} < 0 < \hat{d}_{iS}.$$

If $u_i > 0$ then by (6.8) $u_{i+n} = 0$, so by (6.7),

$$\hat{d}_{iI} < 0 < u_i \leqslant \hat{d}_{iS}.$$

whence $u_i \in (0, \hat{d}_{iS}]$. So by (6.10),

$$u_i \in \{0\} \cup (0, \hat{d}_{iS}] = [0, \hat{d}_{iS}].$$

If $u_{i+n} > 0$ then by (6.9), $u_i = 0$ so by (6.7),

$$\hat{d}_{iI} \leqslant -u_{i+n} < 0 < \hat{d}_{iS},$$

whence $u_{i+n} \in (0, -\hat{d}_{iI}]$. So by (6.10),

$$u_{i+n} \in \{0\} \cup (0, -\hat{d}_{iI}] = [0, -\hat{d}_{iI}].$$

Therefore,

$$(\hat{d}_{iI} < 0 < \hat{d}_{iS}) \Rightarrow (u_i \in [0, \hat{d}_{iS}] \wedge u_{i+n} \in [0, -\hat{d}_{iI}]). \tag{6.14}$$

Finally, suppose that

$$0 \leqslant \hat{d}_{iI}.$$

Then by (6.7),

$$0 \leqslant \hat{d}_{iI} \leqslant u_i - u_{i+n} \leqslant \hat{d}_{iS}. \tag{6.15}$$

If $u_i > 0$ then by (6.8), $u_{i+n} = 0$ so by (6.15),

$$0 \leqslant \hat{d}_{iI} \leqslant u_i \leqslant \hat{d}_{iS},$$

whence if $u_i > 0$ then $u_i \in \hat{\boldsymbol{d}}_i$. *If* $u_{i+n} > 0$ then by (6.9), $u_i = 0$ so by (6.15), $0 \leqslant -u_{i+n}$, whence by (6.10), $u_{i+n} = 0$ contrary to the hypothesis that $u_{i+n} > 0$. So $u_{i+n} = 0$. Therefore,

$$(0 \leqslant \hat{d}_{iI}) \Rightarrow (u_i \in \hat{\boldsymbol{d}}_i \wedge u_{i+n} = 0). \tag{6.16}$$

Therefore the formulae

$$\hat{\boldsymbol{u}}_i = \boldsymbol{0} \quad \text{and} \quad \hat{\boldsymbol{u}}_{i+n} = -\hat{\boldsymbol{d}}_i \quad (\hat{d}_{iS} \leqslant 0), \tag{6.17}$$

$$\hat{\boldsymbol{u}}_i = [0, \hat{d}_{iS}] \quad \text{and} \quad \hat{\boldsymbol{u}}_{i+n} = [0, -\hat{d}_{iI}] \quad (\hat{d}_{iI} < 0 < \hat{d}_{iS}), \tag{6.18}$$

$$\hat{\boldsymbol{u}}_i = \hat{\boldsymbol{d}}_i \quad \text{and} \quad \hat{\boldsymbol{u}}_{i+n} = \boldsymbol{0} \quad (0 \leqslant \hat{d}_{iI}), \tag{6.19}$$

$(i = 1, \ldots, n)$ can be used to determine

$$\hat{\boldsymbol{u}} = (\hat{\boldsymbol{u}}_i, \ldots, \hat{\boldsymbol{u}}_{2n})^{\mathrm{T}},$$

which contain the corresponding Lagrange multipliers.

7. Application

We know that most of the nonlinear programming problems are not in the form (4.1). Therefore, their feasible regions are of irregular shape, whence our algorithm cannot be used directly. However, if we can transform the irregular shape into parallelepiped then the method can be applied.

Suppose that we consider the problem of the form

$$\begin{aligned} &\text{maximize} \quad f(x_1, x_2) \\ &\text{subject to} \quad g(x_1, x_2) \leqslant 0, \end{aligned} \tag{7.1}$$

where $g(x_1, x_2)$ is of the irregular shape. Assume that the feasible region for this problem is simply-connected and its boundary

$$r = u(\theta) \tag{7.2}$$

is a single-valued function of θ as shown in Fig. 1, where

$$r = (x_1^2 + x_2^2)^{1/2}, \tag{7.3}$$

$$\theta = \tan^{-1}(x_2/x_1). \tag{7.4}$$

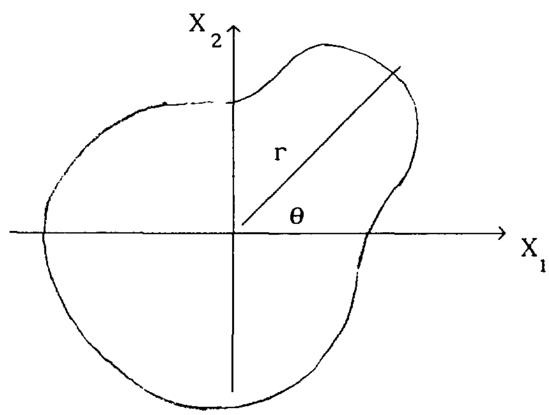

Fig. 1. An irregular region.

By transformation [4], we have

$$x_1 = tu(\theta)\cos\theta,$$
$$x_2 = tu(\theta)\sin\theta.$$

Therefore, the problem becomes

$$\begin{aligned} &\text{maximize} && f(tu(\theta)\cos\theta, tu(\theta)\sin\theta) \\ &\text{subject to} && 0 \leqslant t \leqslant 1, \\ & && 0 \leqslant \theta \leqslant 2\pi. \end{aligned} \tag{7.5}$$

By writing

$$\hat{t} = [0,1] \quad \text{and} \quad \hat{\theta} = [0,2\pi] \tag{7.6}$$

and using (4.2) and (4.3) we obtain the problem

$$\begin{aligned} &\text{maximize} && f(tu(\theta)\cos\theta,\ tu(\theta)\sin\theta) \\ &\text{subject to} && c_i(t,\theta) \geqslant 0 \ \ (i = 1,2,3,4), \end{aligned} \tag{7.7}$$

where

$$c_1(t,\theta) = t - \hat{t}_{\mathrm{I}} = t \geqslant 0,$$
$$c_2(t,\theta) = \theta - \hat{\theta}_{\mathrm{I}} = \theta \geqslant 0,$$
$$c_3(t,\theta) = \hat{t}_{\mathrm{S}} - t = 1 - t \geqslant 0,$$
$$c_4(t,\theta) = \hat{\theta}_{\mathrm{S}} - \theta = 2\pi - \theta \geqslant 0.$$

Therefore, the problems (7.1)–(7.4) can be transformed into the form (4.1) precisely. Now our method can be applied for determining the interval vector

$$\hat{\boldsymbol{u}} = (\hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_2, \hat{\boldsymbol{u}}_3, \hat{\boldsymbol{u}}_4)^{\mathrm{T}},$$

which contains the corresponding Lagrange multipliers.

8. Conclusion

In [9], Robinson states that, in practice, his method can work if one simply chooses an interval vector $\hat{\boldsymbol{z}}$ and tries it [3, 6, 4, 8], and if it does not work, adjusts $\hat{\boldsymbol{z}}$ and tries again.

In this paper, we have shown that when an interval vector $\hat{\boldsymbol{x}}$ has been chosen, the formulae (6.17)–(6.19) can be used to bound the Lagrange multipliers so $\hat{\boldsymbol{z}} = (\hat{\boldsymbol{x}}^{\mathrm{T}}, \hat{\boldsymbol{u}}^{\mathrm{T}})^{\mathrm{T}}$ can be determined and the methods which are given in [3, 6, 4, 8] can be employed for computing and bounding the local minimizer(s) of (4.1). Therefore, we have a method for determining the interval vector which contains the Lagrange multipliers corresponding to the global minimizer(s).

If the feasible region of irregular shape can be transformed into parallelepiped as shown in Section 7 then our method can be employed to the problem safely.

References

[1] G. Alefeld and J. Herzberger, *Introduction to Interval Computations* (Academic Press, New York, 1983).
[2] A.V. Fiacco and G.P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques* (Wiley, London, 1976).
[3] E.R. Hansen, On solving systems of equations using interval arithmetic, *Math. Comp.* **22** (1968) 374–384.
[4] K. Ichida and Y. Fujii, An interval arithmetic method for global optimization, *Computing* **23** (1979) 85–97.
[5] I.B. Mohd, Unconstrained global optimization using strict complementary slackness, *Appl. Math. Comput.* **36** (2) (1990) 75–87.
[6] I.B. Mohd, An interval global optimization algorithm for a class of functions with several variables. *J. Comput. Appl. Math.* **31** (1990) 373–382.
[7] R.E. Moore, *Methods and Applications of Interval Analysis* (SIAM, Philadelphia, PA, 1979).
[8] K.L.E. Nickel, On the Newton method in interval analysis, Tech. Summary Report No. 1136, Mathematics Research Center, University of Wisconsin, Madison, WI, 1971.
[9] S.M. Robinson, Computable error bounds for nonlinear programming, *Math. Programming* **8** (1973) 235–242.