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Analysis on robust stability for interval descriptor systems ☆

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Abstract

This paper considers the robust stability problem for linear continuous-time descriptor systems with interval system matrices. We first derive some necessary conditions for the considered interval descriptor system to be regular, impulse-free and stable. Then, under some constraints, a necessary and sufficient condition is given by using the Kronecker product and μ-analysis. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Descriptor systems present a much wider class of systems than normal systems [2,10,13]. Recently, the robust stability problem for descriptor systems has been addressed (see [3,6,9,11] and the references therein). However, these presented results are mainly on descriptor systems with A-matrix having uncertainties. If the derivative matrix E is perturbed, the problem is quite involved, due to the change of the rank of E, and this problem is seldom studied. In [4], Hu et al. study the robust stability for descriptor systems with E-matrix being subject to unstructured perturbations. In [5], the considered uncertainty of E-matrix is of the unidirectional nature.

For standard state-space systems, studies on the stability for system matrix A being interval have been done (see [12,14] and the references therein). For descriptor systems, the robust stability problem with system matrix A being interval was considered in [8,7], and necessary and sufficient conditions were derived using different methods.

In practice, the structure and behavior of a descriptor system are directly related to the derivative matrix E, so it is meaningful to study uncertain descriptor systems with the matrix E under possible perturbations. Motivated by these observations, we will focus in this paper on the robust stability problem for structured uncertain descriptor systems with the system matrices E and A being subjected to interval perturbations. By

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using the technique of structured singular value and Kronecker product, we give partial solutions for the robust stability. This paper is organized as follows. In Section 2, some existing results together with preliminaries are presented. The main results are given in Section 3. This paper is concluded in Section 4.

2. Problem statement and preliminaries

For given $n \times n$ matrices $A^m = [a_{ij}^m]$ and $A^M = [a_{ij}^M]$ with $a_{ij}^m \leqslant a_{ij}^M$, we use $[A^m, A^M]$ to denote the interval matrix, i.e., the set of all matrices $A = [a_{ij}]$ satisfying $a_{ij}^m \leqslant a_{ij} \leqslant a_{ij}^M$, $i,j = 1,\ldots,n$. We will use $\mu_\Delta(M)$ to denote the structured singular value of M with respect to the set of all allowable Δ. (See [1,15] for mixed μ-analysis.)

Consider the following uncertain linear continuous-time descriptor system

$$E\dot{x}(t) = Ax(t), \tag{1}$$

where $E \in [E^m, E^M]$ and $A \in [A^m, A^M]$ are interval matrices.

For a descriptor system (E, A), if $\det(\alpha E - A) \neq 0$ for some $\alpha \in C$, then it is called *regular*, in which case the existence and uniqueness of the solution of the system will be guaranteed. If $\deg\det(sE - A) = \operatorname{rank} E$, then it is called *impulse-free*. Otherwise, it will possess impulsive modes, which are undesirable in system control. If all the poles of $\det(sE - A)$ lie in the open left-half complex plane, then it is said to be *stable*.

The following lemmas are useful in the development.

Lemma 2.1. *Let I_n and A be $n \times n$ identity matrix and $m \times l$ matrix, respectively. Then we have*

$$\operatorname{rank}[I_n \otimes A \;\; A \otimes I_n] < 2nl, \tag{2}$$

where $\otimes$ denotes the Kronecker product.

Proof. Let the columns of A be $a_i = [a_{1i} \;\cdots\; a_{mi}]^{\mathrm{T}}$, $i = 1,2,\ldots,l$. If $a_i = 0$, then $\operatorname{rank}[I_n \otimes a_i \;\; a_i \otimes I_n] = 0$. If $a_i \neq 0$, without loss of generality, assume that $a_{1i} \neq 0$. We have

$$\operatorname{rank}[I_n \otimes a_i \;\; a_i \otimes I_n] = \operatorname{rank}\begin{bmatrix} a_i & & & a_{1i}I_n \\ & a_i & & a_{2i}I_n \\ & & \ddots & \vdots \\ & & a_i & a_{mi}I_n \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & & & a_{1i}I_n \\ -a_{2i}a_{1i}^{-1}a_i & a_i & & a_{2i}I_n \\ & & \ddots & \vdots \\ -a_{mi}a_{1i}^{-1}a_i & & a_i & a_{mi}I_n \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & & & a_{1i}I_n \\ 0 & a_i & & a_{2i}I_n \\ \vdots & & \ddots & \vdots \\ 0 & & a_i & a_{mi}I_n \end{bmatrix}$$

$$< 2n.$$

Thus, $\operatorname{rank}[I_n \otimes A \;\; A \otimes I_n] \leqslant \sum_{i=1}^{l} \operatorname{rank}[I_n \otimes a_i \;\; a_i \otimes I_n] < 2nl$. This completes the proof. □

Lemma 2.2 ([11]). *Let* $E, A \in \mathbb{R}^{n\times n}$ *and* $\deg\det(sE - A) = \operatorname{rank} E$. *If for any two eigenvalues* λ_1 *and* λ_2 *of the pair* (E,A), *there holds* $\lambda_1 + \lambda_2 \neq 0$, *then*

$$\dim \operatorname{null}(E \otimes A + A \otimes E) = (n - \operatorname{rank} E)^2 \tag{3}$$

where $\operatorname{null}(\cdot)$ *denotes the nullity of* $(\cdot)$.

Lemma 2.2 implies the following result.

Lemma 2.3. *Let* $E, A \in \mathbb{R}^{n\times n}$ *and* $\operatorname{rank} E = r$. *If the pair* (E,A) *is regular, impulse-free and stable, then*

$$\operatorname{rank}(E \otimes A + A \otimes E) = 2nr - r^2. \tag{4}$$

Proof. Note that the stability of the pair (E,A) implies that for any two eigenvalues λ_1 and λ_2 of the pair (E,A), there holds $\lambda_1 + \lambda_2 \neq 0$. Since $\dim(E \otimes A + A \otimes E) = n^2$, the result follows immediately from Lemma 2.2. □

3. Robust stability

For system (1), Let

$$E_0 = \tfrac{1}{2}(E^m + E^M), \quad [E^m, E^M] = E_0 + [\bar{E}^m, \bar{E}^M] \tag{5}$$

and

$$A_0 = \tfrac{1}{2}(A^m + A^M), \quad [A^m, A^M] = A_0 + [\bar{A}^m, \bar{A}^M]. \tag{6}$$

Here, we use $N + [N^m, N^M]$ to denote that each matrix in $[N^m, N^M]$ is added with N. It is obvious that $N + [N^m, N^M] = [N + N^m, N + N^M]$ is still an interval matrix. Then, every entry of the matrix in $[\bar{E}^m, \bar{E}^M]$ and $[\bar{A}^m, \bar{A}^M]$ is either zero or perturbed with the symmetric interval $[\frac{1}{2}(e_{ij}^m - e_{ij}^M), \frac{1}{2}(e_{ij}^M - e_{ij}^m)]$ and $[\frac{1}{2}(a_{ij}^m - a_{ij}^M), \frac{1}{2}(a_{ij}^M - a_{ij}^m)]$, respectively. Denote q and p the number of perturbed entries of E (and thus of $[\bar{E}^m, \bar{E}^M]$) and A (and thus of $[\bar{A}^m, \bar{A}^M]$), respectively. For each perturbed entry in ith row and jth column of $[\bar{E}^m, \bar{E}^M]$ and $[\bar{A}^m, \bar{A}^M]$, define the matrix $\bar{E}_{ij}$ and $\bar{A}_{ij}$ by

$$\bar{E}_{ij} \triangleq \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \frac{1}{2}(e_{ij}^M - e_{ij}^m) & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \qquad \bar{A}_{ij} \triangleq \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \frac{1}{2}(a_{ij}^M - a_{ij}^m) & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{7}$$

Then each of the above matrices contains only one nonzero entry. Label these matrices from E_1 to E_q and from A_1 to A_p, respectively. Then we have

$$[E^m, E^M] = E_0 + \sum_{i=1}^{q} \alpha_i E_i, \quad -1 \leqslant \alpha_i \leqslant 1, \;\; i = 1, \ldots, q, \tag{8}$$

$$[A^m, A^M] = A_0 + \sum_{j=1}^{p} \beta_j A_j, \quad -1 \leqslant \beta_j \leqslant 1, \;\; j = 1, \ldots, p. \tag{9}$$

Suppose $r = \operatorname{rank} E_0$. Then there exist two nonsingular matrices P and Q such that

$$PE_0Q = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \triangleq \bar{E}_0. \tag{10}$$

Define

$$\bar{E}_i \triangleq PE_iQ = \begin{bmatrix} E_{11}^{(i)} & E_{12}^{(i)} \\ E_{21}^{(i)} & E_{22}^{(i)} \end{bmatrix}, \quad i = 1, 2, \ldots, q, \tag{11}$$

$$\bar{A}_j \triangleq PA_jQ = \begin{bmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ A_{21}^{(j)} & A_{22}^{(j)} \end{bmatrix}, \quad j = 0, 1, \ldots, p. \tag{12}$$

With the transformation matrices P and Q, system (1) is restricted equivalent [13] to the following uncertain system:

$$\left(\bar{E}_0 + \sum_{i=1}^{q} \alpha_i \bar{E}_i\right) \dot{x}(t) = \left(\bar{A}_0 + \sum_{j=1}^{p} \beta_j \bar{A}_j\right) x(t), \tag{13}$$

where $-1 \leqslant \alpha_i \leqslant 1$ and $-1 \leqslant \beta_j \leqslant 1$. So, to solve our problem, it suffices to study the robust stability of system (13). The following result provides some necessary conditions for system (13) to be regular, impulse-free and stable.

Theorem 3.1. *If system* (13) *is regular, impulse-free and stable for all* $-1 \leqslant \alpha_i \leqslant 1$ *and* $-1 \leqslant \beta_j \leqslant 1$, *then all the following statements are true*:
(i) $E_{22}^{(i)} = 0$ *for all* $i = 1, 2, \ldots, q$.
(ii) *If* $E_{21}^{(i)} \neq 0$, *then* $E_{12}^{(i)} = 0$. (*If* $E_{12}^{(i)} \neq 0$, *then* $E_{21}^{(i)} = 0$.)
(iii) *If* $E_{21}^{(i)} \neq 0$ *and* $E_{12}^{(j)} \neq 0$, *then* $E_{21}^{(i)}[E_{11}^{(j)} \; E_{12}^{(j)}] = 0$, *and* $\begin{bmatrix} E_{11}^{(i)} \\ E_{21}^{(i)} \end{bmatrix} E_{12}^{(j)} = 0$.

Proof. (i) Suppose for some k, $E_{22}^{(k)} \neq 0$. We prove in the following that system (13) is not regular, impulse-free and stable for all $i = 1, 2, \ldots, q$.

Note that for all $i = 1, 2, \ldots, q$, we have $\operatorname{rank} \bar{E}_i = 1$. Since $E_{22}^{(k)} \neq 0$, it is easy to see that there exist two nonsingular matrices P_k and Q_k such that

$$P_k \bar{E}_0 Q_k = \bar{E}_0, \qquad P_k \bar{E}_k Q_k = \begin{bmatrix} 0_r & & \\ & 1 & \\ & & 0_{n-r-1} \end{bmatrix}, \tag{14}$$

where $0_r \in \mathbb{R}^{r \times r}$ is a zero matrix. Let

$$P_k \bar{A}_0 Q_k = \begin{bmatrix} A_{r+1} & \times \\ \times & A_{n-r-1} \end{bmatrix}, \tag{15}$$

where $A_{r+1} \in \mathbb{R}^{(r+1)\times(r+1)}$ and $A_{n-r-1} \in \mathbb{R}^{(n-r-1)\times(n-r-1)}$. By setting $\alpha_i = 0$ $(i \neq k)$ and $\beta_j = 0$, system (13) is restricted equivalent to the system

$$\left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & \alpha_k & 0 \\ 0 & 0 & 0_{n-r-1} \end{bmatrix}, \begin{bmatrix} A_{r+1} & \times \\ \times & A_{n-r-1} \end{bmatrix} \right). \tag{16}$$

Three cases may occur.

Case 1: If A_{n-r-1} is singular, then system (16) is not impulse-free for all $-1 \leqslant \alpha_k \leqslant 1$. Thus, system (13) and hence system (1) is not regular, impulse-free and stable for all $-1 \leqslant \alpha_i \leqslant 1$.

Case 2: If $\det(P_k \bar{A}_0 Q_k) = 0$, then it is obvious that system (16) is not stable for all $-1 \leqslant \alpha_k \leqslant 1$. Thus, system (13) and hence system (1) is not regular, impulse-free and stable for all $-1 \leqslant \alpha_i \leqslant 1$.

Case 3: If A_{n-r-1} is nonsingular and $\det(P_k\bar{A}_0Q_k) \neq 0$, then the characteristic polynomial of system (16) is given by

$$\begin{aligned}\mathrm{pol}(\lambda,\alpha_k) &= \det\left(\lambda\begin{bmatrix} I_r & 0 & 0\\ 0 & \alpha_k & 0\\ 0 & 0 & 0_{n-r-1}\end{bmatrix} - \begin{bmatrix} A_{r+1} & \times \\ \times & A_{n-r-1}\end{bmatrix}\right)\\ &= \det(A_{n-r-1})\det\left(\begin{bmatrix} I_r & 0\\ 0 & \alpha_k\end{bmatrix}\right)\lambda^{r+1} + \cdots + (-1)^{r+1}\det(P_k\bar{A}_0Q_k)\\ &= \alpha_k\det(A_{n-r-1})\lambda^{r+1} + \cdots + (-1)^{r+1}\det(P_k\bar{A}_0Q_k).\end{aligned} \tag{17}$$

It is easy to see that, for some $\alpha_k \in [-1,1]/\{0\}$, the signs of $\alpha_k\det(A_{n-r-1})$ and $(-1)^{r+1}\det(P_k\bar{A}_0Q_k)$ can be opposite by selecting $\alpha_k > 0$ or $\alpha_k < 0$. From polynomial stability theory, $\mathrm{pol}(\lambda,\alpha_k)=0$ has a root in the right complex half plane, which implies that system (16) is not stable for all $-1 \leqslant \alpha_k \leqslant 1$. Thus, system (13) and hence system (1) is not regular, impulse-free and stable for all $-1 \leqslant \alpha_i \leqslant 1$. This completes the proof of (i).

(ii) Since $E_{22}^{(i)} = 0$, and $\mathrm{rank}\,\bar{E}_i = 1$, then $E_{21}^{(i)} \neq 0$ implies $E_{12}^{(i)} = 0$. The dual case follows similarly.

(iii) With (i) and (ii) in hand, if $E_{21}^{(i)} \neq 0$ and $E_{12}^{(j)} \neq 0$, noticing that $\mathrm{rank}\,\bar{E}_i = \mathrm{rank}\,\bar{E}_j = 1$, we see that there exist two matrices T_L and T_R with appropriate dimensions such that $E_{11}^{(i)} = T_LE_{21}^{(i)}$ and $E_{11}^{(j)} = E_{12}^{(j)}T_R$. Then, by performing row transformation to $E_{21}^{(i)}$ and column transformation to $E_{12}^{(j)}$, we can easily find two nonsingular matrices

$$P_{ij} = \begin{bmatrix} I_r & T_L\\ 0 & I\end{bmatrix} \quad \text{and} \quad Q_{ij} = \begin{bmatrix} I_r & 0\\ T_R & I\end{bmatrix}$$

such that

$$P_{ij}\bar{E}_0Q_{ij} = \bar{E}_0, \quad P_{ij}\bar{E}_iQ_{ij} = P_{ij}\bar{E}_i = \begin{bmatrix} 0_r & 0\\ E_{21}^{(i)} & 0\end{bmatrix}, \quad P_{ij}\bar{E}_jQ_{ij} = \bar{E}_jQ_{ij} = \begin{bmatrix} 0_r & E_{12}^{(j)}\\ 0 & 0\end{bmatrix}. \tag{18}$$

Suppose that $E_{21}^{(i)}E_{12}^{(j)} \neq 0$. Let

$$T_i = \begin{bmatrix} I_r & 0\\ -\alpha_iE_{21}^{(i)} & I\end{bmatrix} \quad \text{and} \quad T_j = \begin{bmatrix} I_r & -\alpha_jE_{12}^{(j)}\\ 0 & I\end{bmatrix}.$$

Using a method similar to that of (i) (the only difference is that $T_iP_{ij}\bar{A}_0Q_{ij}T_j$ is related to α_i and α_j. But this does not affect the proof when α_i and α_j are sufficiently small), it can be proved that the system $(T_iP_{ij}(\bar{E}_0 + \alpha_i\bar{E}_i + \alpha_j\bar{E}_j)Q_{ij}T_j,\ T_iP_{ij}\bar{A}_0Q_{ij}T_j)$ is not impulse-free and stable for all $-1 \leqslant \alpha_k \leqslant 1$ $(k = i, j)$. This is a contradiction. So, it must hold $E_{21}^{(i)}E_{12}^{(j)} = 0$. Noticing that

$$\mathrm{rank}\,[E_{11}^{(j)}\ E_{12}^{(j)}] = \mathrm{rank}\begin{bmatrix} E_{11}^{(i)}\\ E_{21}^{(i)}\end{bmatrix} = 1,$$

we have

$$E_{21}^{(i)}[E_{11}^{(j)}\ E_{12}^{(j)}] = 0 \quad \text{and} \quad \begin{bmatrix} E_{11}^{(i)}\\ E_{21}^{(i)}\end{bmatrix}E_{12}^{(j)} = 0.$$

This completes the proof. □

It is seen from (i) and (ii) in the above theorem that either

$$[E_{21}^{(i)}\ E_{22}^{(i)}] = 0 \quad \text{or} \quad \begin{bmatrix} E_{12}^{(i)}\\ E_{22}^{(i)}\end{bmatrix} = 0$$

holds. In this paper, we consider a special case, for which we will give a necessary and sufficient condition for the robust stability. The special case is based on Assumption 1.

Assumption 1.

$$[E_{21}^{(i)}\ E_{22}^{(i)}]=0 \quad \text{and} \quad \begin{bmatrix} E_{12}^{(i)} \\ E_{22}^{(i)} \end{bmatrix}=0 \quad \text{for all } i=1,2,\ldots,q.$$

The following lemma will be useful to achieve our main result.

Lemma 3.1. *Suppose* $(\bar{E}_0,\bar{A}_0)$ *is regular, impulse-free and stable, and* $\operatorname{rank}(\bar{E}_0+\sum_{i=1}^{q}\alpha_i\bar{E}_i)\equiv r$. *Then system* (13) *is regular, impulse-free and stable for all* $-1\leqslant\alpha_i\leqslant 1$ *and* $-1\leqslant\beta_j\leqslant 1$ *if and only if*

$$\operatorname{rank}\left(\left(\bar{E}_0+\sum_{i=1}^{q}\alpha_i\bar{E}_i\right)\otimes\left(\bar{A}_0+\sum_{j=1}^{p}\beta_j\bar{A}_j\right)+\left(\bar{A}_0+\sum_{j=1}^{p}\beta_j\bar{A}_j\right)\otimes\left(\bar{E}_0+\sum_{i=1}^{q}\alpha_i\bar{E}_i\right)\right)$$
$$=2nr-r^2 \tag{19}$$

holds for all $-1\leqslant\alpha_i\leqslant 1$ *and* $-1\leqslant\beta_j\leqslant 1$.

Proof. The necessity follows easily from Lemma 2.3. For sufficiency, we prove by contradiction. Suppose that system (13) is not regular, impulse-free and stable for all $-1\leqslant\alpha_i\leqslant 1$ and $-1\leqslant\beta_j\leqslant 1$. Denote

$$\mathscr{E}(\alpha)\overset{\triangle}{=}\left\{\bar{E}_0+\sum_{i=1}^{q}\alpha_i\bar{E}_i\,\middle|\,-1\leqslant\alpha_i\leqslant 1\right\},\qquad \mathscr{A}(\beta)\overset{\triangle}{=}\left\{\bar{A}_0+\sum_{j=1}^{p}\beta_j\bar{A}_j\,\middle|\,-1\leqslant\beta_j\leqslant 1\right\}.$$

We consider two cases:

Case 1: There exist $E(\alpha)\in\mathscr{E}(\alpha)$ and $A(\beta)\in\mathscr{A}(\beta)$ such that $(E(\alpha),A(\beta))$ is not regular or not impulse-free. For this case, let two nonsingular matrices T_1 and T_2 render

$$T_1E(\alpha)T_2=\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},\quad T_1A(\beta)T_2=\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}. \tag{20}$$

Then, A_4 is not of full rank. By setting $\operatorname{rank}A_4=w$, we can easily find two nonsingular matrices T_3 and T_4 such that

$$T_3T_1E(\alpha)T_2T_4=\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},\quad T_3T_1A(\beta)T_2T_4=\begin{bmatrix} A_1 & 0 & A_{22} \\ 0 & I_w & 0 \\ A_{32} & 0 & 0 \end{bmatrix}. \tag{21}$$

Hence, we have the following rank equalities:

$$\operatorname{rank}(E(\alpha)\otimes A(\beta)+A(\beta)\otimes E(\alpha))$$
$$=\operatorname{rank}((T_3T_1E(\alpha)T_2T_4)\otimes(T_3T_1A(\beta)T_2T_4)+(T_3T_1A(\beta)T_2T_4)\otimes(T_3T_1E(\alpha)T_2T_4))$$
$$=\operatorname{rank}\begin{bmatrix} A_1\otimes I_r+I_r\otimes A_1 & I_r\otimes[0\ A_{22}] & [0\ A_{22}]\otimes I_r & 0 \\ I_r\otimes\begin{bmatrix} 0 \\ A_{32} \end{bmatrix} & I_r\otimes\begin{bmatrix} I_w & 0 \\ 0 & 0 \end{bmatrix} & 0 & 0 \\ \begin{bmatrix} 0 \\ A_{32} \end{bmatrix}\otimes I_r & 0 & \begin{bmatrix} I_w & 0 \\ 0 & 0 \end{bmatrix}\otimes I_r & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\leqslant r^2+2rw+\operatorname{rank}[I_r\otimes A_{22}\ \ A_{22}\otimes I_r],$$

where the second equality can be referred to [6]. From Lemma 2.1, we obtain

$$\begin{aligned}\operatorname{rank}(E(\alpha)\otimes A(\beta)+A(\beta)\otimes E(\alpha)) &< r^2+2rw+2r(n-r-w)\\ &= 2nr-r^2.\end{aligned}$$

This is a contradiction.

Case 2: System (13) is impulse-free (thus regular) for all $-1\leqslant\alpha_i\leqslant 1$ and $-1\leqslant\beta_j\leqslant 1$. But for some matrix $E(\alpha)\in\mathscr{E}(\alpha)$ and $A(\beta)\in\mathscr{A}(\beta)$, $(E(\alpha),A(\beta))$ is not stable. By assumption, (13) is impulse-free for all $-1\leqslant\alpha_i\leqslant 1$, $-1\leqslant\beta_j\leqslant 1$ and $\operatorname{rank}(\bar{E}_0+\sum_{i=1}^{q}\alpha_i\bar{E}_i)\equiv r$, hence the roots of $\det(s(\bar{E}_0+\sum_{i=1}^{q}\alpha_i\bar{E}_i)-(\bar{A}_0+\sum_{j=1}^{p}\beta_j\bar{A}_j))$ is continuous with respect to α_i and β_j. Noting that $(\bar{E}_0,\bar{A}_0)$ is impulse-free and stable, there must exist some α_i^*'s and β_j^*'s satisfying $-1\leqslant\alpha_i^*\leqslant 1$ and $-1\leqslant\beta_j^*\leqslant 1$, or say $E_*=\bar{E}_0+\sum_{i=1}^{q}\alpha_i^*\bar{E}_i$ and $A_*=\bar{A}_0+\sum_{j=1}^{p}\beta_j^*\bar{A}_j$, such that (E_*,A_*) have imaginary eigenvalues, say $\pm\omega j$ for some $\omega\in\mathbb{R}$. Now, let two nonsingular matrices T_1 and T_2 render

$$T_1E_*T_2=\begin{bmatrix} I_r & 0\\ 0 & 0\end{bmatrix},\qquad T_1A_*T_2=\begin{bmatrix} A_r & 0\\ 0 & I_{n-r}\end{bmatrix} \tag{22}$$

with $\pm\omega j$ being eigenvalues of A_r. Thus, 0 is an eigenvalue of $A_r\oplus A_r=A_r\otimes I_r+I_r\otimes A_r$, that is,

$$\operatorname{rank}(A_r\oplus A_r)<r^2. \tag{23}$$

Then, we check that

$$\begin{aligned}&\operatorname{rank}(E_*\otimes A_*+A_*\otimes E_*)\\ &\quad=\operatorname{rank}((T_1\otimes T_1)(E_*\otimes A_*+A_*\otimes E_*)(T_2\otimes T_2))\\ &\quad=\operatorname{rank}((T_1E_*T_2)\otimes(T_1A_*T_2)+(T_1A_*T_2)\otimes(T_1E_*T_2))\\ &\quad=\operatorname{rank}\begin{bmatrix} A_r\otimes I_r+I_r\otimes A_r & 0 & 0\\ 0 & I_{2nr-2r^2} & 0\\ 0 & 0 & 0\end{bmatrix}\\ &\quad< r^2+(2nr-2r^2)=2nr-r^2\end{aligned} \tag{24}$$

which is also a contradiction. This completes the proof. □

We are now in a position to give the main result.

Theorem 3.2. *Suppose that Assumption* 1 *holds and* $(\bar{E}_0,\bar{A}_0)$ *is regular, impulse-free and stable. Then,* $\operatorname{rank}(\bar{E}_0+\sum_{i=1}^{q}\alpha_i\bar{E}_i)\equiv r$ *and system* (13) *is regular, impulse-free and stable for all* $-1\leqslant\alpha_i\leqslant 1$ *and* $-1\leqslant\beta_j\leqslant 1$ *if and only if any of the following holds*:

$$(1)\ \mu_{\Delta}(H)<1, \tag{25}$$

$$(2)\ \mu_{\bar{\Delta}}(\bar{H})<1, \tag{26}$$

where $H \in \mathbb{R}^{(q+2p)(2nr-r^2)\times(q+2p)(2nr-r^2)}$ *and* $\bar{H} \in \mathbb{R}^{(p+2q)(2nr-r^2)\times(p+2q)(2nr-r^2)}$ *are as follows*:

$$H = \begin{bmatrix} F_1H_0^{-1} & \cdots & F_1H_0^{-1} & F_1H_0^{-1} & -H_{11} & \cdots & F_1H_0^{-1} & -H_{1p} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_qH_0^{-1} & \cdots & F_qH_0^{-1} & F_qH_0^{-1} & -H_{q1} & \cdots & F_qH_0^{-1} & -H_{qp} \\ G_1H_0^{-1} & \cdots & G_1H_0^{-1} & G_1H_0^{-1} & 0 & \cdots & G_1H_0^{-1} & 0 \\ H_0^{-1} & \cdots & H_0^{-1} & H_0^{-1} & 0 & \cdots & H_0^{-1} & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ G_pH_0^{-1} & \cdots & G_pH_0^{-1} & G_pH_0^{-1} & 0 & \cdots & G_pH_0^{-1} & 0 \\ H_0^{-1} & \cdots & H_0^{-1} & H_0^{-1} & 0 & \cdots & H_0^{-1} & 0 \end{bmatrix}, \tag{27}$$

$$\bar{H} = \begin{bmatrix} G_1H_0^{-1} & \cdots & G_1H_0^{-1} & G_1H_0^{-1} & -H_{11} & \cdots & G_1H_0^{-1} & -H_{q1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ G_pH_0^{-1} & \cdots & G_pH_0^{-1} & G_pH_0^{-1} & -H_{1p} & \cdots & G_pH_0^{-1} & -H_{qp} \\ F_1H_0^{-1} & \cdots & F_1H_0^{-1} & F_1H_0^{-1} & 0 & \cdots & F_1H_0^{-1} & 0 \\ H_0^{-1} & \cdots & H_0^{-1} & H_0^{-1} & 0 & \cdots & H_0^{-1} & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_qH_0^{-1} & \cdots & F_qH_0^{-1} & F_qH_0^{-1} & 0 & \cdots & F_qH_0^{-1} & 0 \\ H_0^{-1} & \cdots & H_0^{-1} & H_0^{-1} & 0 & \cdots & H_0^{-1} & 0 \end{bmatrix}, \tag{28}$$

with

$$H_0 = \begin{bmatrix} A_{11}^{(0)}\otimes I_r + I_r\otimes A_{11}^{(0)} & I_r\otimes A_{12}^{(0)} & A_{12}^{(0)}\otimes I_r \\ I_r\otimes A_{21}^{(0)} & I_r\otimes A_{22}^{(0)} & 0 \\ A_{21}^{(0)}\otimes I_r & 0 & A_{22}^{(0)}\otimes I_r \end{bmatrix},$$

$$F_i = \begin{bmatrix} A_{11}^{(0)}\otimes E_{11}^{(i)} + E_{11}^{(i)}\otimes A_{11}^{(0)} & E_{11}^{(i)}\otimes A_{12}^{(0)} & A_{12}^{(0)}\otimes E_{11}^{(i)} \\ E_{11}^{(i)}\otimes A_{21}^{(0)} & E_{11}^{(i)}\otimes A_{22}^{(0)} & 0 \\ A_{21}^{(0)}\otimes E_{11}^{(i)} & 0 & A_{22}^{(0)}\otimes E_{11}^{(i)} \end{bmatrix}, \quad i = 1,2,\dots,q,$$

$$G_j = \begin{bmatrix} A_{11}^{(j)}\otimes I_r + I_r\otimes A_{11}^{(j)} & I_r\otimes A_{12}^{(j)} & A_{12}^{(j)}\otimes I_r \\ I_r\otimes A_{21}^{(j)} & I_r\otimes A_{22}^{(j)} & 0 \\ A_{21}^{(j)}\otimes I_r & 0 & A_{22}^{(j)}\otimes I_r \end{bmatrix}, \quad j = 1,2,\dots,p,$$

$$H_{ij} = \begin{bmatrix} A_{11}^{(j)}\otimes E_{11}^{(i)} + E_{11}^{(i)}\otimes A_{11}^{(j)} & E_{11}^{(i)}\otimes A_{12}^{(j)} & A_{12}^{(j)}\otimes E_{11}^{(i)} \\ E_{11}^{(i)}\otimes A_{21}^{(j)} & E_{11}^{(i)}\otimes A_{22}^{(j)} & 0 \\ A_{21}^{(j)}\otimes E_{11}^{(i)} & 0 & A_{22}^{(j)}\otimes E_{11}^{(i)} \end{bmatrix}, \quad i = 1,\dots,q, \quad j = 1,\dots,p$$

and $\Delta \in \mathbb{R}^{(q+2p)(2nr-r^2)\times(q+2p)(2nr-r^2)}$ *and* $\bar{\Delta} \in \mathbb{R}^{(p+2q)(2nr-r^2)\times(p+2q)(2nr-r^2)}$ *are as follows*:

$$\Delta = \mathrm{diag}\{\alpha_1 I_{(2nr-r^2)},\dots,\alpha_q I_{(2nr-r^2)},\beta_1 I_{2(2nr-r^2)},\dots,\beta_p I_{2(2nr-r^2)}\},$$

$$\bar{\Delta} = \mathrm{diag}\{\beta_1 I_{(2nr-r^2)},\dots,\beta_p I_{(2nr-r^2)},\alpha_1 I_{2(2nr-r^2)},\dots,\alpha_q I_{2(2nr-r^2)}\}.$$

Proof. Since $(\bar{E}_0, \bar{A}_0)$ is regular, impulse-free and stable, from Lemma 2.3, we have

$$\operatorname{rank}(\bar{E}_0 \otimes \bar{A}_0 + \bar{A}_0 \otimes \bar{E}_0) = \operatorname{rank} H_0 = 2nr - r^2$$

i.e., H_0 is invertible. Under Assumption 1, $\operatorname{rank}(\bar{E}_0 + \sum_{i=1}^{q} \alpha_i \bar{E}_i) \leqslant r$. From the proof of Lemma 3.1 (or, refer to [6]), we see that $\operatorname{rank}(B \otimes C + C \otimes B) \leqslant 2n \operatorname{rank} B - (\operatorname{rank} B)^2$ for any $n \times n$ matrices B and C. Thus, if (19) holds, there must hold $\operatorname{rank}(\bar{E}_0 + \sum_{i=1}^{q} \alpha_i \bar{E}_i) = r$. By using Lemma 3.1, we have that, $\operatorname{rank}(\bar{E}_0 + \sum_{i=1}^{q} \alpha_i \bar{E}_i) \equiv r$ and system (13) is regular, impulse-free and stable for all $-1 \leqslant \alpha_i \leqslant 1$ and $-1 \leqslant \beta_j \leqslant 1$ if and only if (19) holds for all $-1 \leqslant \alpha_i \leqslant 1$ and $-1 \leqslant \beta_j \leqslant 1$, which is equivalent to

$$\operatorname{rank}\left(H_0 + \sum_{i=1}^{q} \alpha_i F_i + \sum_{j=1}^{p} \beta_j G_i + \sum_{i=1}^{q} \sum_{j=1}^{p} \alpha_i \beta_j H_{ij}\right) = 2nr - r^2 \tag{29}$$

holds for all $-1 \leqslant \alpha_i \leqslant 1$ and $-1 \leqslant \beta_j \leqslant 1$. Note that (29) holds

$$\Leftrightarrow \det\left(H_0 + \sum_{i=1}^{q} \alpha_i F_i + \sum_{j=1}^{p} \beta_j G_i + \sum_{i=1}^{q} \sum_{j=1}^{p} \alpha_i \beta_j H_{ij}\right) \neq 0$$

$$\Leftrightarrow \det\left(H_0 + \left[\alpha_1 I \ \cdots \ \alpha_q I \quad \beta_1 I \quad \beta_1 \sum_{i=1}^{q} \alpha_i H_{i1} \ \cdots \ \beta_p I \quad \beta_p \sum_{i=1}^{q} \alpha_i H_{ip}\right] \begin{bmatrix} F_1 \\ \vdots \\ F_q \\ G_1 \\ I \\ \vdots \\ G_p \\ I \end{bmatrix}\right) \neq 0$$

$$\Leftrightarrow \det\left(I + \begin{bmatrix} F_1 \\ \vdots \\ F_q \\ G_1 \\ I \\ \vdots \\ G_p \\ I \end{bmatrix} H_0^{-1} \left[\alpha_1 I \ \cdots \ \alpha_q I \quad \beta_1 I \quad \beta_1 \sum_{i=1}^{q} \alpha_i H_{i1} \ \cdots \ \beta_p I \quad \beta_p \sum_{i=1}^{q} \alpha_i H_{ip}\right]\right) \neq 0,$$

where we have used the fact $\det(I + XY) = \det(I + YX)$. Since

$$\left[\alpha_1 I \ \cdots \ \alpha_q I \quad \beta_1 I \quad \beta_1 \sum_{i=1}^{q} \alpha_i H_{i1} \ \cdots \ \beta_p I \quad \beta_p \sum_{i=1}^{q} \alpha_i H_{ip}\right]$$

$$
=[\,\alpha_1 I \;\cdots\; \alpha_q I \quad \beta_1 I \quad 0 \;\cdots\; \beta_p I \quad 0\,]
\begin{bmatrix}
I & & 0 & \beta_1 H_{11} & & 0 & \beta_p H_{1p} \\
& \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\
& & I\;0 & \beta_1 H_{q1} & & 0 & \beta_p H_{qp} \\
& & I & & & & \\
& & & I & & & \\
& & & & \ddots & & \\
& & & & & I & \\
& & & & & & I
\end{bmatrix}
$$

$$
=[\,\alpha_1 I \;\cdots\; \alpha_q I \quad \beta_1 I \quad 0 \;\cdots\; \beta_p I \quad 0\,]
\begin{bmatrix}
I & & 0 & -\beta_1 H_{11} & & 0 & -\beta_p H_{1p} \\
& \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\
& & I\;0 & -\beta_1 H_{q1} & & 0 & -\beta_p H_{qp} \\
& & I & & & & \\
& & & I & & & \\
& & & & \ddots & & \\
& & & & & I & \\
& & & & & & I
\end{bmatrix}^{-1},
$$

the equivalence of (29) is continued as

$$
\det\left(\left(I+
\begin{bmatrix}
0 & & 0 & -\beta_1 H_{11} & & 0 & -\beta_p H_{1p} \\
& \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\
& & 0\;0 & -\beta_1 H_{q1} & & 0 & -\beta_p H_{qp} \\
& & 0 & & & & \\
& & & 0 & & & \\
& & & & \ddots & & \\
& & & & & 0 & \\
& & & & & & 0
\end{bmatrix}\right)\right.
$$

$$
\left. +
\begin{bmatrix}
F_1 \\ \vdots \\ F_q \\ G_1 \\ I \\ \vdots \\ G_p \\ I
\end{bmatrix}
H_0^{-1}[\,\alpha_1 I \;\cdots\; \alpha_q I \quad \beta_1 I \quad 0 \;\cdots\; \beta_p I \quad 0\,]\right) \neq 0
$$

$$\Leftrightarrow \det\left(I + \begin{bmatrix} 0 & & 0 & -H_{11} & & 0 & -H_{1p} \\ & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\ & & 0\ 0 & -H_{q1} & & 0 & -H_{qp} \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \\ & & & & & & 0 \end{bmatrix} \Delta \right.$$

$$\left. + \begin{bmatrix} F_1 \\ \vdots \\ F_q \\ G_1 \\ I \\ \vdots \\ G_p \\ I \end{bmatrix} H_0^{-1} [\, I \ \cdots \ II \quad 0 \ \cdots \ I \quad 0 \,] \Delta \right) \neq 0.$$

$$\Leftrightarrow \det(I + H\Delta) \neq 0.$$

Thus, the result in (25) follows because $\det(I + H\Delta) \neq 0$ for all $-1 \leqslant \alpha_i,\ \beta_j \leqslant 1$ if and only if $\mu_\Delta(H) < 1$.

For (26), note that (29) is also equivalent to

$$\det\left(H_0 + \sum_{i=1}^{q} \alpha_i F_i + \sum_{j=1}^{p} \beta_j G_i + \sum_{i=1}^{q} \sum_{j=1}^{p} \alpha_i \beta_j H_{ij} \right) \neq 0$$

$$\Leftrightarrow \det\left(H_0 + \left[\beta_1 I \ \cdots \ \beta_p I \quad \alpha_1 I \quad \alpha_1 \sum_{j=1}^{p} \beta_j H_{1j} \ \cdots \ \alpha_q I \quad \alpha_q \sum_{j=1}^{p} \beta_j H_{qj} \right] \begin{bmatrix} G_1 \\ \vdots \\ G_p \\ F_1 \\ I \\ \vdots \\ F_q \\ I \end{bmatrix} \right) \neq 0.$$

Hence, the result in (26) follows from a procedure similar to that of (25). This completes the proof of the theorem. □

Remark 3.1. Theorem 3.2 presents two criteria for the robust stability problem. If $q \geqslant p$, we can choose (25) to avoid higher-dimensional computations; otherwise, choose (26).

4. Conclusion

This paper studies the robust stability problem for linear continuous-time interval descriptor systems. It first derives some necessary conditions for the system to be regular, impulse-free and stable. Then, under some constraint on the uncertainties of the derivative matrix, a necessary and sufficient condition is obtained. As for the general case (i.e., no constraint on the derivative matrix), the problem is quite involved and still remains open.

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