Computational Mathematics and Modeling, Vol. 5, No. 1, 1994

CONTROL OF NONLINEAR AND UNCERTAIN SYSTEMS
MODAL CONTROL UNDER INTERVAL UNCERTAINTY OF PARAMETERS

N. A. Khlebalin

1. INTRODUCTION

How to allow for uncertain parameter values in the analysis and synthesis of automatic systems has always been a central problem in control theory. The method of log frequency characteristics handles uncertain parameters indirectly, by introducing stability margins in absolute value and in phase. In the framework of sensitivity and invariance theory, polar uncertainties are considered – infinitesimally small or infinitely large. The robustness theory, which emerged about a decade ago, focuses on methods that allow for finite dispersion of the parameters; no restrictions are imposed on the shape of the uncertainty region in the parameter space. Another direction that emerged at about the same time considers a special type of parametric uncertainty – interval uncertainty, which frequently occurs in practice.

One of the first studies in the analysis of interval systems is [1]. Sufficient stability conditions for automatic systems under interval uncertainty of parameters were developed in [2]. Controllers in linear automatic systems are synthesized by two methods: analytic design or modal control. The analytic design method solves a nonlinear matrix Riccati equation. Its application to the case of interval uncertainty in the matrix elements of the plant model has produced a solution of the synthesis problem in a restricted setting (see [3, 4] and elsewhere). The modal control method, on the other hand, essentially reduces to solving a system of linear algebraic equations and uses other techniques from linear algebra. A method based on modal control for the synthesis of controllers with interval uncertainty of parameters was developed in [5] using the apparatus of interval analysis [6].

In this paper, we consider two approaches to modal controller synthesis under interval uncertainty of parameters (UUP). The first approach specifies the desired interval polynomial and uses the controller to include in its coefficients the coefficients of the characteristic polynomial of a closed-loop system (the closed-loop characteristic polynomial, or CLCP) with uncertain parameters. The second approach does not require specification of a desired interval polynomial. It uses the effect of contraction of the CLCP interval coefficients around their desired real values. The efficiency of the proposed approaches is analyzed in application to the interval optimal control problem.

2. STATEMENT OF THE PROBLEM

Consider the system

$$\dot{x}(t)=[A]\cdot x(t)+[b]\cdot u(t), \tag{1}$$

where x is the n-dimensional state vector, u is the scalar control generated by the law

$$u(t)=k\cdot x(t), \tag{2}$$

[A], [b], k are appropriately dimensioned matrices and vectors, t is time. The matrix [A] and the vector [b] are interval constructs [6], i.e.,

$$a_{ij}\in[\underline{a}_{ij},\overline{a}_{ij}],\ b_i\in[\underline{b}_i,\overline{b}_i]\ (i,\ j=\overline{1,n}), \tag{3}$$

Translated from Upravlenie Nelineinymi Sistemami, Vsesoyuznyi Nauchno-Issledovatel'skii Institut Sistemnykh Issledovanii, Sbornik Trudov, No. 4, pp. 50-59, 1991.

 1046-283X/94/0501-0044$12.50

where $\underline{a}_{ij}$, $\underline{b}_i$ are the lower bounds and $\bar{a}_{ij}$, $\bar{b}_i$ the upper bounds of the intervals in which the parameters take their allowed values.

It is required to a find a vector k from (2) such that the closed-loop system (1)-(3) has the desired modal characteristics.

Let us elucidate the last statement. Modal control usually specifies the desired distribution of the poles of the closed-loop system or the desired CLCP. If the parameter values are known exactly, both methods of specifying the modal characteristics are equivalent. Under interval uncertainty of parameters, the modal characteristics may diverge. The conditions on the closed-loop system, as we shall see, can be defined in the form of an ordinary polynomial with real coefficients, an interval polynomial, an interval polynomial with linearly dependent coefficients, a signomial interval polynomial, a simply connected (trapezoid-shaped) or multiply connected root-localization region containing the CLCP roots, or an ordinary spectrum in the complex plane. To obtain the required modal characteristic, we have to apply an individual synthesis method for the vector k.

3. SYNTHESIS METHOD FOR A MODAL CONTROLLER WITH EXACTLY KNOWN PARAMETERS [7]

In this case, the system has the form

$$\dot{x}(t)=A\cdot x(t)+b\cdot u(t),$$

and its transfer function is

$$W(s)=(I\cdot s-A)^{-1}\cdot b=adj(I\cdot s-A)\cdot b/d(s),$$

where $d(s) = \det(I \cdot s - A)$ is the determinant of the matrix $I \cdot s - A$, or the plant characteristic polynomial (PCP), I is the $n \times n$ identity matrix, $\mathrm{adj}(I \cdot s - A)$ is the adjoint matrix. The CLCP can be written in the form

$$-k\cdot \mathrm{adj}(I\cdot s-A)\cdot b+d(s)=0. \tag{4}$$

Equating the coefficients of the polynomial (4) to the coefficients of equal powers of s in the desired polynomial, we obtain (because of linearity in the elements of the vector k) the system of algebraic equations

$$\begin{aligned} &p_{11}(a_{ij},b_j)\cdot k_1+\ldots+p_{1n}(a_{ij},b_i)\cdot k_n+d_1(a_{ij})=\Delta_1^* \\ &\cdots \\ &p_{n1}(a_{ij},b_i)\cdot k_1+\ldots+p_{nn}(a_{ij},b_i)\cdot k_n+d_n(a_{ij})=\Delta_n^*, \end{aligned} \tag{5}$$

where $p_{11}(\cdot), \ldots, p_{nn}(\cdot), d_1(\cdot), \ldots, d_n(\cdot)$ are functions of appropriate arguments, and $\Delta_1^*, \ldots, \Delta_n^*$ are the coefficients of the desired polynomial. The system (5) can be written in matrix form and solved as follows:

$$P\cdot k^T+d=\Delta^*,\ P\cdot k^T=\Delta^*-d,\ P\cdot k^T=h,\ k^T=P^{-1}\cdot h, \tag{6}$$

where $P = \| p_{\gamma\rho}(a_{ij}, b_i) \|$ is an $n \times n$ matrix, $d = \| d_\gamma(a_{ij}) \|$ is the n-dimensional vector of PCP coefficients, Δ^* is the n-dimensional vector of coefficients of the desired polynomial, T denotes the transpose, $\gamma, \rho = 1, \ldots, n$.

The inverse P^{-1} exists if the plant is completely controllable.

4. SOLUTION OF THE INTERVAL CONTROL PROBLEM BY THE INCLUSION METHOD

The proposed approach, first described in [5], relies on the property of inclusionwise monotonicity of interval polynomials [8]. According to this property, if the roots of some interval polynomial

$$[\Delta^*(s)]=s^n+[\Delta_1^*]\cdot s^{n-1}+\ldots+[\Delta_{n-1}^*]\cdot s+[\Delta_n^*] \tag{7}$$

are localized in the region Ω^*, then the roots of any other interval polynomial of the same degree

$$[\Delta(s)] = s^n + [\Delta_1]\cdot s^{n-1} + \ldots + [\Delta_{n-1}]\cdot s + [\Delta_n] \tag{8}$$

are localized in the region $\Omega \subseteq \Omega^*$ (here $\subseteq$ stands for set-theoretical nonstrict inclusion) if

$$[\Delta_1]\subseteq[\Delta_1^*], \ldots, [\Delta_{n-1}]\subseteq[\Delta_{n-1}^*], \quad [\Delta_n]\subseteq[\Delta_n^*]. \tag{9}$$

In particular, if the polynomial $[\Delta^*(s)]$ is stable (Ω^* is entirely in the left half of the complex plane), then $[\Delta(s)]$ is also stable.

The vector k is computed by the following four-stage algorithm.

Stage 1. Given is the desired interval characteristic polynomial (7). Its coefficients are required to satisfy "length" constraints, which are specialized in stage 3.

Stage 2. Compute the intervals for the elements of the matrix P and the vector d from (6), i.e., represent the matrix and the vector in interval form [P] and [d].

In this context, let us consider the properties of the coefficients of the CLCP (4).

Definition. The expression $\sum_{i=1}^{m} \sigma_i \cdot q_i \cdot \prod_{j=1}^{\mathscr{V}} \nu_j^{\theta_j}$, where $\sigma_i = \pm 1$, q_i are arbitrary positive real numbers, and ν_j are real variables (parameters), is called a signomial [9] of first degree if $\theta_j = 1$ ($j = \overline{1, \mathscr{V}}$).

Proposition 1. The coefficients of the CLCP (4) are first-degree signomials in the parameters a_{ij}, b_i, (i, j = 1, ..., n).

The proof is given in the Appendix.

Proposition 2. If the parameters ν_j ($j=\overline{1,\mathscr{V}}$) in a first-degree signomial vary within some interval $[\nu_j]$, then the first-degree signomial attains its greatest and least value at some vertices of the interval vector (hyperprism) $\|[\nu_1], [\nu_2], \ldots, [\nu_{\mathscr{V}}]\|$.

The proof is given in the Appendix.

Propositions 1 and 2 enable us to compute the exact bounds of the elements of the matrix P and the vector d from (6). To this end, we have to enumerate all possible combinations of the interval bounds from (3) (the corner points of the hyperprism of uncertain parameters), to compute P and d at each corner point (a suitable FORTRAN program is given in [10]), and to find the greatest and least values of the elements of [P] and [d].

Stage 3. Compute the elements of the interval vector [h] from (6) by the formula

$$\underline{h}_i = \underline{\Delta}_i^* - \underline{d}_i, \quad \overline{h}_i = \overline{\Delta}_i^* - \overline{d}_i \quad (i = \overline{1, n}). \tag{10}$$

This formula represents the operation of "location by summation" introduced in [5], and it does not fit in the framework of "traditional" interval arithmetic [6]. Informally (10) means that the length $\mathscr{W}([\Delta_i^*]) = \bar{\Delta}_i^* - \Delta_i^*$ of each coefficient of the polynomial (7) should be not less than the length of the corresponding CLCP coefficient. On the other hand, these lengths are upper bounded by the stability conditions and therefore it is not always possible to construct (7) for every plant (1), (3).

Step 4. The vector k is computed so as to ensure the inclusion

$$[P]\cdot k^T \subseteq [h]. \tag{11}$$

This, however, is not always achievable. The simplest way to obtain k that satisfies (11) is by solving "centers–lengths" systems [5]. Denote by m([P]) and m([h]) the matrix and the vector formed from the centers of the elements of [P] and [h], and by $\mathscr{W}$([P]) and $\mathscr{W}$([h]) the matrix and the vector formed from the lengths of the elements of [P] and [h].

Proposition 3. The vector k that ensures inclusion of the coefficients of the characteristic polynomial of the closed-loop system (1), (2) in the coefficients of the desired interval CLCP (7) for any values of the uncertain parameters from (3) is the solution of the system m([P])·m([h]) subject to the vector-matrix inequality $\mathscr{W}([P]) \cdot |k^T| \leqslant \mathscr{W}([h])$.

The proof is given in [5]. In general, k is determined by solving mathematical-programming problems [11, 12]. Thus, for instance, the vector k is an element of the guaranteed solution of the system of linear interval algebraic equations (SLIAE) $[P]\cdot[k] \subseteq [h]$, which is obtained from at most 2^n linear programming problems of the form [11]

$$\sum_{i=1}^{n}(y_i-y_{n+i})\to\max;\ F\cdot y\leqslant\varphi;$$

$$F=\left\|\begin{array}{c:c}-1 & I\\ \hdashline \bar{\Lambda} & 0\\ \hdashline 0 & -\underline{\Lambda}\end{array}\right\|;\quad \varphi=\left\|\begin{array}{c}0\\ \hdashline \bar{h}\\ \hdashline -\underline{h}\end{array}\right\|;$$

where

$$y=\|y_1,\ldots,y_n,y_{n+1},\ldots,y_{2n}\|^T,\ y_i=\bar{k}_i(i=\overline{1,n}),\quad y_{n+i}=k_i(i=\overline{1,n}).$$

If the signs of the elements of [k] are known, then the matrices $\underline{\Lambda}$ and $\bar{\Lambda}$ are formed from the bounds of the intervals [P] in such a way that $\Lambda\cdot k\le\underline{h}$, $\bar{\Lambda}\cdot\bar{k}\ge h$. Theoretically, this information is not available a priori and we therefore have to solve the 2^n problems written above, because this is the number of different sign combinations of the elements of an n-dimensional vector. We pass to the next problem when one of the components of the previous solution is zero; the transition involves interchanging the columns in the matrices $\underline{\Lambda}$ and $\bar{\Lambda}$ that correspond to zero elements k_i ($i = 1, \ldots, n$). In technical problems, individual controller coefficients change their sign fairly seldom, and the signs of the elements of the vector k therefore can be determined by solving an ordinary modal control problem for a specific combination of plant parameter values from (3).

The property of inclusionwise monotonicity of interval polynomials enables us to specify the modal characteristics in the interval modal control problem in the form of a simply connected region that contains the poles of the closed-loop system [13]. The algorithm that solves this problem is iterative: given the desired region Ω^* it constructs an interval polynomial (7) which is consistent with the constraints of stage 3 and simplifies the computations in stage 4 of the previous algorithm [14]. The solution of the problem can be sought as a real or an interval vector (a controller with tolerances) [13, 15].

The ordinary modal control problem is solvable if the plant is completely controllable. For plants with parametric uncertainty, controllability criteria are insufficiently developed [13]. We can only assert that if the controllability matrix of an interval plant (1), (3)

$$M_y=\|[b],\ [A]\cdot[b],\ [A]^2\cdot[b],\ldots,[A]^{n-1}\cdot[b]\|\tag{12}$$

is of rank n, then for any parameter values from (3) the pair (A, b) is completely controllable [13, 16]. From a practical point of view, the character of this criterion (necessary and sufficient, only sufficient, and its "magnitude") depends on the particular interval extension [8] that can be constructed for the determinant of the $n\times n$ matrix formed from the columns of the controllability matrix (12) taken in an arbitrary order. If the interval extension is joint, the criterion preserves its necessity and sufficiency. If this cannot be proved, then the criterion is only sufficient. Note that for the interval modal control problem complete controllability of the plant is not a sufficient condition of solvability in the framework of the proposed approach. Thus, we can give examples when (11) is unsolvable with a nonsingular [P].

The proposed approach has been implemented by the FORTRAN program XSRMMY. Some modifications of the program are proposed in [17, 18, 19].

5. SOLUTION OF THE INTERVAL MODAL CONTROL PROBLEM BY THE CONTRACTION METHOD

The main idea of the inclusion method is to find a feedback controller which locates the CLCP coefficients inside the coefficients of the desired interval polynomial. First, however, we have to construct an interval polynomial with sufficiently "wide" coefficients. Experience with solution of practical problems [20] has shown that this is very difficult for multidimensional plants with $n\ge 5$. The point is that the problem of construction of the desired interval polynomial, i.e., determination of the intervals of its coefficient given the localization region of the set of its roots, has a nonunique solution [14]. Moreover, as it follows from Proposition 1, the CLCP coefficients are not interval (independent) coefficients and the interval-polynomial representation of the CLCP imposes over-restrictive conditions on the system. For these reasons, the problem is often unsolvable.

Here we define system performance in the usual way (in the form of the desired spectrum or in the form of real coefficients of the desired characteristic polynomial) and find the controller which ensures that the bounds of the CLCP interval coefficients deviate by the smallest possible amount from the desired real values, while bracketing the latter.

Consider the interval inclusion $[P] \cdot k^T \supseteq h^*$. We call k the contracting real solution (CRS) of this inclusion. The term "contracting solution" is chosen because it groups (contracts) the elements of each row in the interval matrix around the corresponding element of the right-hand side vector. The CRS, like other solutions of systems of linear interval algebraic equations, does not always exist. Thus, for instance, if the matrix [P] contains a zero row and the corresponding element in the vector h^* is nonzero, then a CRS does not exist. Take an arbitrary real vector k and substitute it in the above inclusion. The result is an n-dimensional interval vector [h] with lower and upper bounds, which we denote $\underline{h} = \underline{[P]\cdot k^T}$, $\bar{h} = \overline{[P]\cdot k^T}$. By definition, k is a CRS if we have the two-sided inequality $\underline{h} \le h^* \le \bar{h}$. If a CRS exists, then it is obviously nonunique, because the "distance" between the inequality bounds may be arbitrarily large. We stipulate that the bounds of the product $[P] \cdot k^T$ are maximally (in some sense) compressed around h^*. This CRS is called optimal (OCRS).

The problem of finding the OCRS is a vector optimization problem, because it requires simultaneous minimization of the lengths of n intervals. To obtain a scalar criterion, we use the following simple technique: multiply the length $\mathcal{W}([h])$ of each i-th element of the vector [h] by a positive weight c_i and add up the products. We thus obtain a linear programming (LP) problem

$$c^T \cdot \mathcal{W}([h]) \to \min \text{ over } k;\ \underline{h} \leqslant h^* \leqslant \bar{h}, \tag{13}$$

because the elements of the vectors $\mathcal{W}([h])$, $\underline{h}$, $\bar{h}$ linearly depend on $k_1, k_2, \ldots, k_n$. This method of computing the CRS does not require any assumptions about the properties of the matrix [P] (nonsingularity, spectral norm constraints, etc.), which are inevitable in existing methods of finding the fixed point of an SLIAE or localizing the inverse [21]. The solution of the LP problem (13) is called a linearly optimal CRS (LOCRS).

Using the notion of LOCRS, we can solve the interval modal control problem by the following proposition.

Proposition 4. Given the signs of the coefficients of the sought vector k in the control law (2), the absolute values χ_j ($j = 1, \ldots, n$) of its coefficients are obtained by solving the following LP problem:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} (\bar{\pi}_{ij} - \underline{\pi}_{ij}) \cdot \chi_j \to \min \text{ over } \chi_j (j = \overline{1, n}), \tag{14}$$

$$\sum_{j=1}^{n} \underline{\pi}_{ij} \cdot \chi_j \leqslant \Delta_i^* - \underline{d}_i,\ (i = \overline{1, n}), \tag{15}$$

$$\sum_{j=1}^{n} \bar{\pi}_{ij} \cdot \chi_j \geqslant \Delta_i^* - \bar{d}_i,\ (i = \overline{1, n}), \tag{16}$$

where

$$\chi_j \geqslant 0,\ \underline{\pi}_{ij} = \underline{p}_{ij},\ \bar{\pi}_{ij} = \bar{p}_{ij}, \text{ if } k_j > 0 (i, j = \overline{1, n})$$
$$\underline{\pi}_{ij} = -\bar{p}_{ij},\ \bar{\pi}_{ij} = -\underline{p}_{ij}, \text{ if } k_j < 0 (i, j = \overline{1, n}), \tag{17}$$

and $\underline{p}_{ij}$, $\underline{d}_i$ and $\bar{p}_{ij}$, $\bar{d}_i$ are the lower and upper bounds of the elements of the matrix [P] and the vector [d].

The proof is given in the Appendix.

The algorithm that solves the interval modal control problem on the basis of Proposition 4 is described below.

Stage 1. Specify the desired polynomial of degree n with real coefficients, i.e., the vector Δ^*.

Stage 2. Use Propositions 1 and 2 to compute [P] and [d].

Stage 3. Write system (5) in the form

$$[P] \cdot k^T + [d] \supseteq \Delta^*. \tag{18}$$

For an arbitrary combination of parameter values from (3), solve the ordinary modal control problem by the algorithm of Sec. 3. Determine the signs of the controller coefficients (the vector k).

Stage 4. Use Proposition 4 to determine the absolute values of the elements of k. If some absolute values are zero, then construct a new LP problem (14)-(17) and find its solution after reversing the signs of the corresponding controller coefficients.

The last stage demonstrates possible nonuniqueness of the solution of this approach. Let us consider the specifics.

First, the LP problem (14)-(17) is not always solvable. Indeed, the existence of real bounding functions (A.1) implies that we can always choose k_j of sufficiently large absolute value so that $\Delta_i \leq \Delta_i^* \leq \bar{\Delta}_i$ ($i = \overline{1, n}$). This means that, contrary to the "inclusion" approach, the present approach always computes k for the control law (2) with any intervals (3). It minimizes the absolute value of k_j, which is very important in applied problems. There is no need to analyze controllability either.

Second, solution of the interval modal control problem by the "contraction" approach does not guarantee stability of the closed-loop system, because this requirement is not incorporated in the LP problem. This fact can be utilized to select stabilizing controllers in case of nonunique termination of stage 4. To ensure that the CLCP (4) has the desired properties (stability, desired region of root localization, etc.) for any values of the uncertain plant parameters from given intervals, it is sufficient to solve the nonlinear programming problem which is obtained by adjoining to (14)-(17) the corresponding conditions (for instance, Hurwitz stability) expressed in terms of the real bounding functions (A.1). In this case, the criterion (14) can be replaced with a nonlinear criterion (minimizing the volume, and not the perimeter, of the hyperprism formed from the CLCP coefficients). This implies passage to a procedure that seeks a nonlinearly optimal CRS of the system of linear interval algebraic equations.

Third, stage 2 implies that the interval-functional matrix (a matrix whose elements are functions of interval parameters) and the interval-functional vector are "coarsened" by embedding in an interval matrix and an interval vector [21]. On the one hand, this simplifies the solution of the problem (it is easy to construct (A.1)), while on the other hand, some properties of the plant are lost and, for instance, it is more difficult to ensure stability of this "coarse" system than in cases that allow for the structure of P and d. Progress in this direction can be achieved because the dependence of the elements of P and d on the plant parameters is established by Proposition 1.

The proposed method has been implemented by the FORTRAN program XSRMY. Solution of the problem from [20] with more restrictive conditions on the localization region of the poles of the closed-loop system has confirmed the efficiency of the method.

6. CONCLUSION

Two approaches have been considered to the solution of the interval modal control problem. The inclusion method is effective when the number of uncertain parameters is small, the magnitude of uncertainty is small, and the plant dimension is small. The contraction method is not sensitive to these characteristics of the problem, but it does not guarantee the desired properties of the system in the class of linear mathematical programming problems.

APPENDIX

Proof of Proposition 1. Directly from the definition of the determinant of a matrix [22] we see that it is a first-degree signomial of the matrix elements. Since the determinant of the sum of two arbitrary real $n \times n$ matrices C and F is the sum of the determinants of these matrices [23], and also the sum of the determinants obtained by replacing in all possible ways $g = 1, \ldots, n - 1$ columns of the first matrix with the corresponding columns of the second matrix, we see that $\det(s \cdot C + F)$ is a polynomial in s with coefficients in the form of first-order signomials of the elements of these matrices. Then, setting $C = I$, $F = -A$, we obtain that the coefficients d(s) are first-order signomials of the parameters a_{ij}. The adjoint matrix is formed from the cofactors of order $n - 1$ of each element of the matrix $s \cdot I - A$, and since the elements of the matrix A and the vectors b and k are independent, the first term in (4) is a polynomial in the variable s of degree $n - 1$ with coefficients in the form of first-order signomials of the parameters a_{ij}, b_i ($i, j = 1, \ldots, n$). These coefficients are linear functions of k_j [7]. Summing the coefficients of the given polynomial and d(s) for equal powers of s, we obtain the sought result. Q.E.D.

Proof of Proposition 2. Assume the contrary: the first-order signomial has an extremum, but one of its variables is inside its interval $\underline{\nu}_l < \nu_l < \bar{\nu}_l$. The first-order signomial by definition is a linear function in this variable: $S(\nu_l) = f \cdot \nu_l + r$, where f and r are computed from σ_i, q_i, $\underline{\nu}_j$, $\bar{\nu}_j$ ($i, \overline{1, m}$; $j = 1, 2, .., l - 1, l + 1, \ldots, \vartheta$). A necessary condition of extremum of this function is that f vanishes. But if $f = 0$, this means that S is independent of ν_l, which is impossible because

ν_l is one of its variables. Therefore f $\neq$ 0 and the derivative $\partial S(\nu_l)/\partial \nu_l$ has a definite sign. For a function monotone in ν_l (and $S(\nu_l)$ is such a function) this means that ν_l can be varied in the corresponding direction and $S(\nu_l)$ will take a greater (or a smaller) value than that initially assumed. This contradicts our starting assumption that the first-order signomial has an extremum. Therefore, the first-order signomial attains its extremum only for $\nu_l = \underline{\nu}_l$ or for $\nu_l = \overline{\nu}_l$. Q.E.D.

Proof of Proposition 4. If the signs of k_j (j = 1, ..., n) are known, for each i-th row in the left-hand side of (18) we can write the real bounding functions [8]

$$\begin{aligned}\underline{\Delta}_i &= \underline{\pi}_{i1}\cdot k_1 + \underline{\pi}_{i2}\cdot k_2 + \ldots + \underline{\pi}_{in}\cdot k_n + \underline{d}_i,\\ \overline{\Delta}_i &= \overline{\pi}_{i1}\cdot k_1 + \overline{\pi}_{in}\cdot k_2 + \ldots + \overline{\pi}_{in}\cdot k_n + \overline{d}_i,\end{aligned} \tag{A.1}$$

where by linearity of the interval-valued functions in the real vector argument k

$$\begin{aligned}&\underline{\pi}_{ij} = \underline{p}_{ij},\ \overline{\pi}_{ij} = \overline{p}_{ij}, \quad \text{if} \quad k_j > 0,\\ &\underline{\pi}_{ij} = -\overline{p}_{ij},\ \overline{\pi}_{ij} = -\underline{p}_{ij}, \quad \text{if} \quad k_j < 0, \ (i, j = \overline{1, n}).\end{aligned} \tag{A.2}$$

The length of the interval of the i-th coefficient $[\Delta_i]$ of the CLCP (4) is expressible in terms of the real bounding functions in the form

$$w([\Delta_i]) = \overline{\Delta}_i - \underline{\Delta}_i = (\overline{\pi}_{i1} - \underline{\pi}_{i1})\cdot k_1 + \ldots (\overline{\pi}_{in} - \underline{\pi}_{in})\cdot k_n + \overline{d}_i - \underline{d}_i.$$

Adding these expressions and omitting the total length of the coefficients of the plant characteristic polynomial (since it does not affect the solution of the LP problem), we obtain (14). Expressions (15), (16) recast in a form suitable for the LP problem represent the condition $\Delta_i^* \in [\Delta_i] = [\underline{\Delta}_i, \overline{\Delta}_i]$. To solve the LP problem, we have to eliminate negative k_j. This can be done by substituting

$$[\underline{p}_{ij}, \overline{p}_{ij}]\cdot k_j = [-\overline{p}_{ij}, -\underline{p}_{ij}]\cdot \chi_j, \quad \text{if} \quad \chi_j > 0. \tag{A.3}$$

Equality (A.3) can be verified directly. Using (A.3), we replace (A.2) with (17). Minimizing the objective function (14) subject to (15), (16), we obtain the sought absolute values of the coefficients of the vector k from the construction of the LP problem. Q.E.D.

REFERENCES

1. S. Faedo, "Un nuovo problema di stabilita per le equazioni algebriche a coefficienti reali," Ann. Scuola Nom. Super. Piza, Sci. Fis. Mat., **7**, No. 1-2, 53-63 (1953).
2. V. L. Kharitonov, "Asymptotic stability of equilibria of a family of systems of linear differential equations," Diff. Uravn., **14**, No. 11, 2086-2088 (1978).
3. S. S. Chang and T. K. S. Peng, "Adaptive guaranteed cost control of systems with uncertain parameters," IEEE Trans. Autom. Contr., **AC-17**, No. 4, 474-483 (1972).
4. I. R. Petersen and M. R. Pickering, "Riccati equations in the stabilization of uncertain systems: a comparison between the methods of Chang–Peng and Petersen–Hollot," in: Preprints of 11th IFAC World Congr., Tallin (1990), pp. 317-321.
5. N. A. Khlebalin, "Analytical method of controller synthesis with uncertain plant parameters," in: Analytical Methods of Controller Synthesis [in Russian], Inter-Univ. Proc., Saratov. Politekh. Inst., Saratov (1981), pp. 107-123.
6. R. E. Moore, Interval Analysis, Prentice-Hall, Englewood Cliffs, N.J. (1966).
7. N. T. Kuzovkov, Aircraft Stabilization Systems [in Russian], Vysshaya Shkola, Moscow (1976).
8. Yu. I. Shokin, Interval Analysis [in Russian], Nauka, Novosibirsk (1981).
9. G. Reklaitis, A. Ravindran, and K. Ragsdell, Engineering Optimization [Russian translation], Vol. 2, Mir, Moscow (1986).
10. J. L. Melsa and S. K. Jones, Computer Programs for Computational Assistance in the Study of Linear Control Systems [Russian translation], Mashinostroenie, Moscow (1981).

11. N. A. Khlebalin, "Guaranteed solution of systems of linear interval algebraic equations," in: Distributed Information-Control Systems [in Russian], Izd. Saratov. Univ., Saratov (1988), p. 138.
12. S. P. Sharyi, "On some solution methods for the linear tolerance problem," Preprint No. 6, VTs SO Akad. Nauk SSSR, Krasnoyarsk (1989).
13. N. A. Khlebalin, "Analysis of controllability-observability and controller synthesis for stationary multidimensional continuous plants in the presence of uncertain parametric perturbations," in: Development of Methods of Analytical Synthesis of Controllers for Multidimensional Systems [in Russian] (1983), pp. 56-64.
14. N. A. Khlebalin, "Construction of interval polynomials with a given root localization region," in: Analytical Methods of Controller Synthesis [in Russian], Inter-Univ. Proc. Saratov. Politekhn. Inst., Saratov (1982), pp. 92-98.
15. N. A. Khlebalin, "Synthesis of interval controllers in the modal control problem," in: Analytical Methods of Controller Synthesis [in Russian], Inter-Univ. Proc. Saratov. Politekhn. Inst., Saratov (1988), pp. 26-30.
16. N. A. Khlebalin, "Qualitative assessment of controllability of plants described by differential equations with uncertain coefficients," in: 6th All-Union Conf. on Qualitative Theory of Differential Equations, abstracts of papers [in Russian], Irkutsk (1986), pp. 193-195.
17. A. V. Zakharov and Yu. I. Shokin, "Synthesis of control systems in the presence of interval uncertainty of parameters in their mathematical models," Dokl. Akad. Nauk SSSR, **299**, No. 2, 292-295 (1988).
18. E. M. Smagina and I. V. Dugarova, "Synthesis of a modal controller for a system with uncertain parameters," Tomsk (1987). Unpublished manuscript, VINITI No. 789-V87.
19. R. T. Sirazetdinov, "Synthesis of the set of controls guaranteeing the desired performance of linear stationary systems with uncertain parameters," in: 11th All-Union Conf. on Control Problems, abstracts of papers [in Russian], Tashkent (1989), pp. 23-24.
20. A. K. Samorukov, N. A. Khlebalin, and A. V. Zakharov, "Controller synthesis for stabilization of lateral motion of a helicopter towing system in the presence of uncertain parameters in the mathematical models," in: Analytical Methods of Controller Synthesis [in Russian], Inter-Univ. Proc. Saratov. Politekhn. Inst., Saratov (1982), pp. 109-122.
21. G. Alefeld and J. Herzberger, Introduction to Interval Computations, Academic Press, New York (1983).
22. G. Korn and T. Korn, Mathematical Handbook for Scientists and Engineers, McGraw-Hill, New York (1961).
23. V. P. Sigorskii, The Mathematical Apparatus of Engineers [in Russian], Tekhnika, Kiev (1977).