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Interval Analysis and Design of Robust Pole Assignment Controllers [1]

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Abstract

The problem of synthesizing robust pole placement controllers for interval linear time-invariant plants is addressed in the context of the concepts and methods of interval analysis. The robust pole placement design problem is associated with the solution of an interval Diophantine equation, whose basic properties are analyzed. Robust pole placement controllers are viewed as *inner solutions* of the interval Diophantine equation. Simple and computationally efficient characterizations of the set of all robust pole placement controllers are then obtained and some of its geometric properties, discussed. Several aspects of the design of robust controllers by interval analysis are integrated into a linear goal programming formulation, which can also incorporate additional constraints on the controller parameters. Robust pole placement design problems are easily modeled and solved through the approach proposed. Numerical examples illustrate its main characteristics.

Key Words - Interval analysis, robust control, pole placement, linear programming.

1 Introduction

POLE placement is a traditional design technique for linear time-invariant control systems. The technique is based on the fact that several performance specifications can be met using dynamic output feedback to adequately place closed-loop poles in the complex plane. An extension of the classical pole placement problem is the regional pole placement problem, in which the objective is to place closed-loop poles in a suitable region of the complex plane. The regional pole placement problem is usually treated in connection with the substantially more general problem of placing closed-loop poles in a specified region in the face of uncertainty with respect to the mathematical model of the plant. In many real-world situations, the model uncertainty reflects on the parameters of the plant, which has motivated extensive research efforts in parametric robust control theory ([1], [3], [5]).

In this paper, uncertain plants are represented by proper transfer functions with coefficients belonging to real intervals and the robust pole assignment problem is formulated as the problem of robustly assign poles in a region determined by the roots of an interval characteristic polynomial. Similar conditions hold in [16], [15] and [7]. In [16], after explicitly characterizing the set of all robust pole placement controllers, the distance of a nominal controller to this set is minimized using a nonlinear programming technique. A sensitivity method applied to a nominal characteristic polynomial is used to specify the target interval characteristic polynomial. In [15], the distance to a nominal controller and the size of the uncertainty region are compromised in a robust pole assignment design technique based on semi-infinite optimization. In [7], some important developments of the last fifteen years in the area of robust control are employed to handle the robust pole assignment problem in a linear programming framework. In particular, the Edge Theorem [4] is used to obtain *root spaces* of interval polynomials, allowing the specification of the worst case performance of the closed-loop system in terms of common frequency domain criteria as, for example, damping ratio and stability degree [5].

The starting point for all the above mentioned robust pole assignment techniques is the classical pole placement design technique. As comprehensively discussed in [2] and [6], among others, the solution of the classical pole placement problem can be reduced, under appropriate conditions, to the solution of the so-called Diophantine equation, whose matrix version assumes the form of a linear system, $Ax = b$, where A is the Sylvester matrix associated with a given n-order plant, x is the vector of coefficients of a r-order controller to be designed and b is the vector of coefficients of a prescribed $(n + r)$-degree characteristic polynomial.

In this paper, it is shown that the analysis and design of robust pole assignment controllers can be systematized through concepts and methods of interval analysis [11]. An *interval* Diophantine equation is derived and its matrix version assumes the form of a linear interval system $A^I x = b^I$, where A^I and b^I are the interval Sylvester matrix and the interval vector

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associated with a given interval plant and with a prescribed interval characteristic polynomial, respectively. The concept and the basic properties of interval Diophantine equations are discussed in Section 2. The one-to-one correspondence between a robust pole assignment controller and an *inner solution* of $A^I x = b^I$ is discussed in Section 3. Simple and computationally efficient characterizations of the set of all robust controllers are introduced and some of its geometric properties are investigated. In Section 4, several aspects of the design of robust controllers by interval analysis are integrated into a goal programming formulation. Robust pole assignment problems are easily modeled and solved through the interval analysis/goal programming approach proposed, which can also incorporate additional constraints on the controller parameters. General conclusions are presented in Section 5.

Notation. Given a real matrix $A = \{a_{ij}\}$, A^T denotes the transpose of A and $|A|$ is defined as $|A| := \{|a_{ij}|\}$. The n-order identity and zero matrices are denoted by I_n and 0_n, respectively. Given two matrices A, B of same size, then $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for each i, j. An interval matrix is defined as $A^I = [A^-, A^+] := \{A : A^- \leq A \leq A^+\}$. The *center* and *radius* matrices of A^I are given by $A_c := (A^+ + A^-)/2$ and $\Delta := (A^+ - A^-)/2$ ($\Delta \geq 0$), respectively. According to this notation, $A^- = A_c - \Delta$, $A^+ = A_c + \Delta$ and $A^I = [A_c - \Delta, A_c + \Delta]$. Similar definitions apply for an interval column vector. In particular, $b^I = [b^-, b^+] = [b_c - \delta, b_c + \delta]$.

2 The interval Diophantine equation

Consider the unit feedback (SISO, time-invariant) control system represented in Figure 1. The transfer functions of a given n-order plant $P(s)$ to be controlled and a series r-order controller $C(s)$ to be designed are represented as

$$P(s) := \frac{n_P(s)}{d_P(s)} \quad \text{and} \quad C(s) := \frac{n_C(s)}{d_C(s)}, \qquad (1)$$

where

$$\begin{aligned} n_P(s) &:= a_1 s^n + a_2 s^{n-1} + \cdots + a_{n+1}, \\ d_P(s) &:= a_{n+2} s^n + a_{n+3} s^{n-1} + \cdots + a_{2n+2}, \\ n_C(s) &:= x_1 s^r + x_2 s^{r-1} + \cdots + x_{r+1}, \\ d_C(s) &:= x_{r+2} s^r + x_{r+3} s^{r-1} + \cdots + x_{2r+2}. \end{aligned}$$

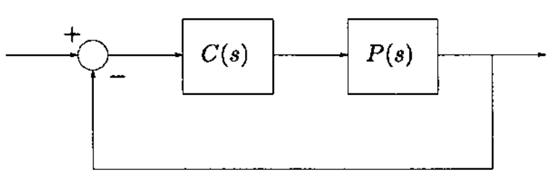

Fig. 1: Unity-feedback SISO system.

The coefficients of $C(s)$ must be selected in order to meet performance specifications translated into pole locations for the closed-loop system. When the coefficients of $P(s)$ are known quantities, classical pole placement analysis ([6], [2]) shows that there exists a solution $x = (x_1, x_2, \ldots, x_{2r+2})$ for the Diophantine equation

$$d_P(s) d_C(s) + n_P(s) n_C(s) = d_F(s) \qquad (2)$$

for all possible $d_F(s)$, where $d_F(s) := b_1 s^{n+r} + b_2 s^{n+r-1} + \cdots + b_{n+r+1}$, if and only if $r \geq n-1$ and $n_P(s)$ and $d_P(s)$ are coprime. For convenience, define $m := n + r + 1$ and $q := 2r + 2$. It is also known that (2) can be written as a linear system,

$$Ax = b, \qquad (3)$$

where

$$A := \begin{bmatrix} a_1 & & & a_{n+2} & & \\ & \ddots & & & \ddots & \\ a_2 & & a_1 & a_{n+3} & & a_{n+2} \\ \vdots & \ddots & & \vdots & \ddots & \\ a_{n+1} & & a_2 & a_{2n+2} & & a_{n+3} \\ & \ddots & \vdots & & \ddots & \vdots \\ & & a_{n+1} & & & a_{2n+2} \end{bmatrix},$$

$$x := \begin{bmatrix} x_1 & x_2 & \cdots & x_q \end{bmatrix}^T, \quad x \in \mathbf{R}^q,$$

and

$$b := \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix}^T, \quad b \in \mathbf{R}^m.$$

Matrix $A \in \mathbf{R}^{m \times q}$ in (3) is the Sylvester matrix associated with the plant. Since any plant, controller or characteristic polynomial is uniquely specified by A, x or b, they will be sometimes referred as *plant* A, *controller* x and *characteristic polynomial* b, respectively.

Assume that the plant coefficients $a = (a_1, a_2, \ldots, a_{2n+2})$ are now uncertain quantities described by closed intervals $a_i^I := [a_i^-, a_i^+]$, $i = 1, 2, \ldots, 2n+2$ and define $a^I := [a^-, a^+]$. Note that the uncertainty description adopted does not assume symmetry with respect to nominal plant coefficients. A coefficient a_i^I will be called *properly uncertain* if $a_i^- < a_i^+$. Plant uncertainties of interval nature give rise to an *interval* Sylvester matrix $A^I := [A^-, A^+]$, whose lower (A^-) and upper (A^+) bounds are obtained when the coefficients of A are replaced by their lower (a^-) and upper (a^+) values, respectively. In addition, assume that the desired region for closed-loop poles is overbounded by $\mathcal{R}(b^I)$, the *root space* of an interval characteristic polynomial $b^I := [b^-, b^+]$. The root space of b^I is the set composed of the roots of all possible polynomials in b^I. The Edge

Theorem [4] establishes that $\mathcal{R}(b^I)$ can be effectively obtained restricting the computations to the edges of b^I.

In a context where both the plant and the characteristic polynomial are specified by intervals, it seems reasonable to address the robust pole assignment problem starting from an *interval* Diophantine equation, $A^I x = b^I$, where A^I and b^I are the interval plant and the interval characteristic polynomial, respectively. The *solution set* of interval equations of the form $A^I x = b^I$ is defined as [14]

$$\Sigma := \{x \ : \ Ax = b \quad \text{for some } A \in A^I \text{ and some } b \in b^I\}.$$

It is interesting to observe that Σ describes the set of all controllers for which there exists a plant $A \in A^I$ and a characteristic polynomial $b \in b^I$. In an integrated plant and controller design context, the knowledge of Σ would allow the joint selection of A, x and b [10]. The following characterization of Σ in terms of center and radius values of A^I and b^I was introduced in [12].

Theorem 1 (Oettli-Prager inequality) *Let $A^I x = b^I$ be a given interval equation, where $A^I = [A_c - \Delta, A_c + \Delta]$ and $b^I = [b_c - \delta, b_c + \delta]$. Then*

$$\Sigma = \{x \ : \ |A_c x - b_c| - \Delta\,|x| \le \delta\}.$$

The main source of difficulties connected with computing the solutions of $Ax = b$ is the complicated (generally non-convex) character of Σ. The problem of developing numerical methods for computing bounds on Σ has been extensively treated in the interval analysis literature [14]. Fortunately, robust pole assignment controllers are related to a subset of Σ that admits much simpler representations, and for which even the order of the controller does not play any significant role.

3 Robust controllers as inner solutions of $A^I x = b^I$

This section is dedicated to the characterization and analysis of robust controllers as inner solutions of interval Diophantine equations. A vector x is called an *inner solution* of the interval linear system $A^I x = b^I$ if $Ax \in b^I$ for all $A \in A^I$ [13]. The set of all inner solutions of $A^I x = b^I$ will be denoted as

$$\Sigma_0 := \{x \ : \ Ax \in b^I \quad \text{for all } A \in A^I\}.$$

The properties of Σ_0 are of particular interest in robust pole assignment control by the obvious reason that Σ_0 is precisely the set of all robust pole assignment controllers. An important consequence of dealing with the robust pole assignment problem in an interval analysis framework is the existence of explicit representations for Σ_0.

Theorem 2 (Representations of Σ_0) *Let Σ_0 be the set of all inner solutions of the linear interval equation $A^I x = b^I$ and define*

$$a)\quad \Sigma_1 := \{x \ : \ |A_c x - b_c| + \Delta\,|x| \le \delta\};$$

$$b)\quad \Sigma_2 := \{x \ : \ x = y - z,\ A^- y - A^+ z \ge b^-,\ A^+ y - A^- z \le b^+,\ y \ge 0,\ z \ge 0\};$$

$$c)\quad \Sigma_3 := \{(x,y) \ : \ A_c x - \Delta\, y \ge b^-,\ A_c x + \Delta\, y \le b^+,\ -y \le x \le y\}.$$

Then $\Sigma_0 = \Sigma_1 = \Sigma_2$ and $x \in \Sigma_0$ if and only if there exists a y such that $(x,y) \in \Sigma_3$.

Proof: See [13] for the proofs involving the equivalences $\Sigma_0 = \Sigma_1 = \Sigma_2$. The correspondence between Σ_0 and Σ_3 is demonstrated in [9]. □

The similarities between Σ (Theorem 1) and Σ_1 (Theorem 3) are apparent. In particular, $\Sigma_1 \subset \Sigma$, as expected from the definition of Σ_1 (that is, Σ_0) as the set of inner solutions of Σ. The first equivalence in Theorem 3, $\Sigma_0 = \Sigma_1$, can be viewed as an interval analysis derivation of the necessary and sufficient condition for robust pole placement established in [16]. A subtle difference is that in [16], the equivalent notations for Δ and δ describe deviations from a *nominal plant* A_c and a *nominal characteristic polynomial* b_c, while in Theorem 3, Δ and δ are simply computable quantities from A^-, A^+, b^- and b^+. In fact, the second equivalence of Theorem 3, $\Sigma_0 = \Sigma_2$, shows that nominal values are not essential to describe robust controllers. Note finally that the equivalence $\Sigma_0 = \Sigma_2$ and the correspondence between Σ_0 and Σ_3 are specially suitable to numerical manipulations.

Several useful geometric properties related to Σ_0 and its equivalent representations can be analyzed and interpreted in terms of robust pole assignment design. The following Propositions are immediate consequences of Theorem 3.

Proposition 1 (Convexity) *Σ_0 is a convex polyhedral set.*

The convex polyhedral nature of Σ_0 (as the set of inner solutions) is made evident in [13] through the equivalence $\Sigma_0 = \Sigma_2$. The convexity of Σ_0 (as the set of robust controllers) is pointed out in [16].

Proposition 2 (Nonemptyness) *Σ_0 is a nonempty set if and only if the linear inequality systems indicated in Σ_2 or Σ_3 have a solution.*

The nonemptyness of Σ_0 can be tested by Phase I of the Simplex algorithm when applied to the linear inequalities in Σ_2 or Σ_3. When Σ_0 is nonempty, one can proceed with the selection of a robust controller satisfying a given criterion, for example. It should be noted that the emptyness of Σ_0 would be caused by an incompatibility between the interval description of the plant, $A^I = [A_c - \Delta, A_c + \Delta]$, and the interval characteristic polynomial specification $b^I = [b^-, b^+]$. Assuming that there exists a x such that $A_c x \in b^I$ and that Δ is *scaled* by a nonnegative factor σ, the following relationship between the magnitude of the plant uncertainty and the existence of robust controllers can be established.

Proposition 3 (Maximal uncertainty) *Consider the nonlinear programming problem in the variables x, y and $\sigma \geq 0$:*

$$\begin{aligned} \max\ & \sigma \\ \text{s.t.}\ & A_c x - (\sigma\Delta)\, y \geq b^-, \\ & A_c x + (\sigma\Delta)\, y \leq b^+, \\ & -y \leq x \leq y. \end{aligned}$$

Let σ^* be the optimal value of σ. Then Σ_0 is nonempty if and only if $\sigma^* \geq 1$ and the maximal uncertainty interval containing a robust controller is $[A_c - \sigma^*\Delta, A_c + \sigma^*\Delta]$.

Proposition 4 (Boundedness) *Σ_0 is bounded if at least one coefficient of $n_P(s)$ and one coefficient of $d_P(s)$ are properly uncertain.*

Proof: Note that $x \in \Sigma_0$ implies $\Delta\,|x| \leq \delta$ (because $x \in \Sigma_1$). An unbounded x can not verify $\Delta\,|x| \leq \delta$ if each column of $\Delta \geq 0$ has at least one element not equal to zero. Since Δ is also a Sylvester matrix, exhibiting the structure indicated in (3), one concludes that Σ_0 is bounded if at least one coefficient of $n_P(s)$ and one coefficient of $d_P(s)$ are properly uncertain. □

Proposition 5 (Bounds on Σ_0) *Suppose that Σ_0 is a nonempty bounded set. For each $i = 1, 2, \ldots, q$, define x_i^- as the minimal value of the linear program*

$$\min_x \ x_i \quad s.\ t. \quad x \in \Sigma_2.$$

Define x_i^+, $i = 1, 2, \ldots, q$ similarly, replacing min *by* max. *Then the least interval vector x^I containing Σ_0 is given by $x^I := [x^-, x^+]$.*

Note that the knowledge of x^I completely determines the range of the coefficients of all possible robust controllers and constitutes an useful design information. In fact, let x^I be partitioned as

$$x^I := \begin{bmatrix} x_z^I \\ \cdots \\ x_p^I \end{bmatrix},$$

where x_z^I and x_p^I denote interval vectors of same dimension, $r+1$. Then the zeros and poles of all possible robust pole assignment controllers will be overbounded by the root spaces of the interval polynomials defined by x_z^I and x_p^I, respectively. A numerical example is used to illustrate this and all the previous results related to Σ_0.

Example 1 - ([16], [15]) Consider the uncertain second order plant

$$P(s) = \frac{s + a_3}{s^2 - 2.2s + a_6},$$

where $a_3 \in [0.5, 1.5]$ and $a_6 \in [-2.6, -2.2]$. It is required to design a first order controller ($r = 1$) to robustly assign closed-loop poles in the root space determined by the interval $b^I = [b^-, b^+]$, where $b^- = (1.000, 9.600, 12.81, 5.120)$ and $b^+ = (1.000, 14.40, 30.17, 24.68)$. The interval Sylvester matrix $A^I = [A^-, A^+]$ is easily computed, as well as A_c, Δ, b_c and δ. There exists a robust pole assignment controller (Σ_0 is nonempty): Phase I of the Simplex method (Proposition 2) applied to Σ_2 provides

$$C(s) = \frac{13.83s + 6.256}{s - 1.852}.$$

An additional investigation through Proposition 3 shows that $\sigma^* = 1.319$, implying that the maximal uncertainty interval for the plant can be made about 32% greater than the initially specified. Since $P(s)$ satisfies the sufficient condition for boundedness (Proposition 4), it is possible to obtain the interval vector x^I using Proposition 5:

$$x^I = [x^-, x^+] = \begin{bmatrix} x_z^I \\ \cdots \\ x_p^I \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 10.13, 16.96 \\ -3.979, 18.89 \end{bmatrix} \\ \cdots\cdots\cdots \\ \begin{bmatrix} 1.000, 1.000 \\ -5.040, 1.664 \end{bmatrix} \end{bmatrix}$$

The root spaces of x_z^I and x_p^I are easily obtained in this particular example. After computing the real roots of the four edges of x_z^I and of the single edge of x_p^I, one concludes that the zeros and poles of all robust controllers are contained in the intervals $[-1.865, 0.393]$ and $[-1.664, 5.040]$, respectively. The closed-loop poles can be robustly assigned by stable or unstable controllers, with minimum phase or non-minimum phase zeros. Of course, at this point it is not possible to assure the existence of a stable, minimum phase robust controller. □

4 A linear goal programming approach

In this section, the existence and design of robust pole assignment controllers will be simultaneously addressed in a goal programming framework ([17]). Goal programming techniques are usually treated in connection with multiple objective optimization problems and are characterized by the introduction of negative (η) and positive (ρ) deviation variables from a goal, a vector of desirable values for the objectives. In the goal programming approach proposed, $A^I x$ plays the role of an interval multiple objective linear function and b^I represents an interval goal to be pursued. In this sense, the robust pole assignment problem can be classified as an interval linear goal programming problem with interval coefficients [8].

Recall that by Theorem 2, a vector x is an element of the set of robust controllers Σ_0 if and only if $x \in \Sigma_2$. Vectors $x = y - z$ and $\xi := [y^T \vdots z^T]^T$ are related by $x = T\xi$, where $T := [I_q \vdots -I_q]$, and any additional specification $x \in \mathcal{X}$ on the controller coefficients can be transformed into an equivalent specification $\xi \in \Xi$ on the coefficients of ξ.

Theorem 3 *Consider the linear goal programming problem in the nonnegative variables ξ, η^-, ρ^+, η^- and ρ^+:*

$$\text{(GP)} \qquad \min \sum_{i=1}^{m} (\eta_i^- + \rho_i^+)$$

$$\text{s.t.} \quad M^-\xi + \eta^- - \rho^- = b^-, \quad M^+\xi + \eta^+ - \rho^+ = b^+, \quad \xi \in \Xi,$$

where

$$M^- = \begin{bmatrix} A^- & \vdots & -A^+ \end{bmatrix}, M^+ = \begin{bmatrix} A^+ & \vdots & -A^- \end{bmatrix},$$

and Ξ is a nonempty polyhedral set. Let v^* be the optimal value of problem (GP), ξ^* the optimal value of ξ and $x^* = T\xi^*$. Then $x^* \in \Sigma_0 \cap \mathcal{X}$ if and only if $v^* = 0$.

Proof: Let $(\xi^*, \eta^{-*}, \rho^{-*}, \eta^{+*}, \rho^{+*})$ be any optimal solution of problem (GP). Note that optimality and the equivalence between Ξ and $\mathcal{X}$ imply that $x^* = T\xi^* \in \mathcal{X}$. It remains to show that $x^* \in \Sigma_0$ if and only if $v^* = 0$. (Sufficiency) If $v^* = 0$, then $\eta^{-*} = \rho^{+*} = 0$, and one obtains

$$A^-y^* - A^+z^* \geq b^-, \quad A^+y^* - A^-z^* \leq b^+, \quad y^* \geq 0,\ z^* \geq 0.$$

The above inequalities have a solution (y^*, z^*) and, according to Theorem 2, $x^* \in \Sigma_0$. (Necessity) Any optimal solution of (GP) is such that $\eta_i^{-*}\rho_i^{-*} = 0$ ($\eta_i^{+*}\rho_i^{+*} = 0$) for all i (see [17]). If $v^* > 0$, then at least one component of η^{-*} or ρ^{+*} is positive and the associated component of ρ^{-*} or η^{+*} is zero. As a consequence, at least one of the above inequalities is violated and, again by Theorem 2, $x^* \notin \Sigma_0$. □

Problem (GP) always has a feasible solution and its optimal value v^* measures the overall deviation of $x^* \in \mathcal{X}$ from Σ_0. If $v^* > 0$, then η^{-*} and ρ^{+*} indicate which components of the interval goal $b^I = [b^-, b^+]$ can not be achieved. In goal programming, it is common to revise goals based on the informations provided by the deviation variables. When $v^* = 0$ ($\eta^{-*} = \rho^{+*} = 0$), the deviation variables η^{+*} and ρ^{-*} act as slack and surplus variables for the inequalities describing robust controllers. If their values are all positive, the robust controller assigns characteristic polynomials strictly inside the proper interval components of $b^I = [b^-, b^+]$. The robust controllers designed by goal programming in Example 2 presents such property.

Specifications related to the structure of the robust controllers introduce constraints on x. A popular specification is that the robust controller must exhibit a PID structure,

$$C_{PID}(s) = \frac{k_D s^2 + k_P s + k_I}{s},$$

where k_D, k_P, k_I are the derivative, proportional and integral gains. It is common to impose bounds on their values: $k_D \in [k_D^-, k_D^+]$, $k_P \in [k_P^-, k_P^+]$, $k_I \in [k_I^-, k_I^+]$. Comparing $C_{PID}(s)$ with $C(s)$ defined in (1) for $r = 2$, it is readily seen that a PID specification requires inequality constraints on x_1, x_2, x_3 and equality constraints on x_4, x_5, x_6. Clearly, $r = 2$ is an artifice for characterizing a PID controller and not its *order*. The corresponding PID specification on ξ would be $\xi \in \Xi_{PID} := \{\xi \ : \ G\xi \leq h,\ E\xi = f\}$, where

$$G := \begin{bmatrix} -I_3 & \vdots & 0_3 & \vdots & I_3 & \vdots & 0_3 \\ \cdots & & \cdots & & \cdots & & \cdots \\ I_3 & \vdots & 0_3 & \vdots & -I_3 & \vdots & 0_3 \end{bmatrix},$$

$$E := \begin{bmatrix} 0_3 & \vdots & I_3 & \vdots & 0_3 & \vdots & -I_3 \end{bmatrix},$$

$$h := \begin{bmatrix} -k^- \\ \cdots \\ k^+ \end{bmatrix}, \ f := \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T,$$

$$k^- := \begin{bmatrix} k_D^- & k_P^- & k_I^- \end{bmatrix}^T, k^+ := \begin{bmatrix} k_D^+ & k_P^+ & k_I^+ \end{bmatrix}^T.$$

Example 2 - The PID robust control of the third order plant

$$P(s) = \frac{a_3 s + a_4}{s^3 + a_6 s^2 + a_7 s + a_8},$$

is considered in [7]. All uncertain coefficients lie on the interval [0.99, 1.01] and it is required to robustly assign closed-loop poles in the root space of $b^I = [b^-, b^+]$, where $b^- = (1.000, 6.000, 31.25, 57.00, 38.25)$ and $b^+ = (1.000, 14.00, 45.25, 77.00, 54.25)$. The PID gains are restricted to the intervals $k_D \in [6, 14]$, $k_P \in [11, 19]$ and $k_I \in [31, 39]$. To account for the condition $r = 2$ and $x_4 = 0$, the components of b^- and b^+ are shifted to the right and zeros are included in the first positions of augmented vectors b^- and b^+. Then letting $\Xi = \Xi_{PID}$, the optimal solution of problem (GP) furnishes $v^* = 0$, and the corresponding robust PID controller

$$C_{PID}(s) = \frac{12.43s^2 + 18.76s + 38.87}{s}.$$

In the linear programming approach proposed in [7], the number of linear inequalities that results from a combinatorial operation with all the vertices of $a^I = [a^-, a^+]$ increases exponentially with n and r. The interval analysis approach proposed in this paper shows that only two vertices, a^- and a^+, are necessary to formulate robust pole assignment problems. The number of linear inequalities in the goal programming formulation (without additional constraints) is $2m$ and increases linearly with n and r. □

5 Conclusions

The most important conclusion of this paper is that the analysis and design of robust pole assignment controllers can be systematized using interval analysis theory. The characterization of the set of robust controllers as inner solutions of an interval Diophantine equation has created conditions to derive several useful properties related to the robust pole assignment problem. As consequence of the use of interval analysis concepts and their interpretation in terms of robust pole assignment, the design of robust controllers has been carried out using a linear goal programming approach, which combines the ease of implementation and efficiency of linear optimization algorithms with an adequate representation of the decision-making aspects involved. An extension of the approach proposed to the case where the interval parameters appear multilinearly in the plant transfer function and a sensitivity analysis (*fragility*) of the robust controllers designed by interval goal programming are under current investigation.

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