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Asymptotic Stability of Interval Time-Delay Systems

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Abstract. In this paper we consider the asymptotic stability of linear interval time-delay systems on the base of using Lyapunov's direct method and methods of interval analysis. A sufficient condition of asymptotic stability is obtained using the concept of Lyapunov–Krasovsky functional.

1. Introduction

A classical theory of stability was considered the last century and allows to investigate a rather wide spectrum of processes when having an exact mathematical description. However, when constructing mathematical models of processes in practice tolerances in model parameters may appear. One way to take into account these tolerances is to specify intervals with known bounds. Mathematical models of such processes can be represented using rules and terminology of intensively developing interval mathematics.

First formulated in [4] and then studied and comprehensively solved in [6] the investigation of asymptotic stability of a characteristic interval polynomial gave an impetus to further research in this field. A further development of the theory represented in [6] generalizes results for the case of interval quasipolynomials [8], occurring when investigating differential time-delay equations. Among the subsequent researches considering the asymptotic stability of the equilibrium position for differential time-delay equations one should mention [7] and the references there. In that paper, which is a development of the theory represented in [8], some sufficient conditions of the asymptotic stability of an interval quasipolynomial were obtained on the base of four functions.

The field of dynamic properties of interval systems given in the state space is somewhat different. Attempts to generalize the results of the paper [6] and to obtain some analogues of Kharitonov's theorems for interval matrices [2] failed. Nowadays there are a lot of papers considering the stability of linear interval systems given in the state space. Among these papers the most fruitful one is the paper by Rohn [11]. There are very few papers devoted to dynamic properties of interval differential time-delay equations given in the state space.

2. Notations and Statement of the Problem

Throughout the remaining part of the paper the bold font will denote interval values, whereas the usual font will denote real (i.e., non-interval) ones. Underlining and overlining an interval will denote the lower and upper bounds of an interval, respectively; mid $\mathbf{a} = (\underline{\mathbf{a}} + \overline{\mathbf{a}}) / 2$ is the midpoint of an interval $\mathbf{a}$; rad $\mathbf{a} = (\overline{\mathbf{a}} - \underline{\mathbf{a}}) / 2$ is the radius of an interval $\mathbf{a}$. The operations mid, rad, taking lower and upper bounds of intervals will be understood componentwise when applied to vectors and matrices.

We consider the interval time-delay system, of which the mathematical model is given in the state space by a differential time-delay equation in the following vector-matrix representation

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{A}_\tau x(t - \tau), \qquad x(\theta) = \varphi_{t_0\tau}(\theta), \;\; \theta \in [t_0 - \tau; t_0], \tag{2.1}$$

where $t \in [t_0 - \tau; \infty)$ is the independent variable (time); $t_0 \in \mathbb{R}$ is the initial time; $\tau \in \mathbb{R}_+$ is the delay; $x(t)$ is the state vector, $x(t) = (x_i(t))$, $x_i(t)$ are continuous functions on $[t_0 - \tau; \infty)$, $1 \leq i \leq n$; $\mathbf{A}, \mathbf{A}_\tau \in \mathbb{IR}^{n\times n}$ are given interval matrices, $\mathbf{A} = (\mathbf{a}_{ij})$, $\mathbf{a}_{ij} = [\underline{\mathbf{a}}_{ij}, \overline{\mathbf{a}}_{ij}]$; $\mathbf{A}_\tau = (\mathbf{a}_{\tau_{ij}})$, $\mathbf{a}_{\tau_{ij}} = [\underline{\mathbf{a}}_{\tau_{ij}}, \overline{\mathbf{a}}_{\tau_{ij}}]$, $1 \leq i, j \leq n$; $\mathbb{IR}$ is the set of all real intervals [1], [15]; $\varphi_{t_0\tau} : [t_0 - \tau; t_0] \to \mathbb{R}^n$ is the initial vector valued function belonging to the space $C[t_0 - \tau; t_0]$ of continuous vector valued functions defined on $[t_0 - \tau; t_0]$.

Throughout this paper a mathematical model of the system is a family of mathematical models of the systems

$$\dot{x}(t) = Ax(t) + A_\tau x(t - \tau), \qquad x(\theta) = \varphi_{t_0\tau}(\theta), \;\; \theta \in [t_0 - \tau; t_0], \tag{2.2}$$

where $A \in \mathbf{A}$ and $A_\tau \in \mathbf{A}_\tau$. In a formal view all just said above will be written down as

$$\begin{aligned}\{x(t) \in \mathbb{R}^n \;\mid\; & \dot{x}(t) = Ax(t) + A_\tau x(t - \tau),\\ & x(\theta) = \varphi_{t_0\tau}(\theta),\; \theta \in [t_0 - \tau; t_0],\; A \in \mathbf{A},\; A_\tau \in \mathbf{A}_\tau\}.\end{aligned} \tag{2.3}$$

DEFINITION 2.1. The interval time-delay system (2.1) is called asymptotically stable, if any system (2.2), where $A \in \mathbf{A}$ and $A_\tau \in \mathbf{A}_\tau$ is asymptotically stable, i.e., for any matrices $A \in \mathbf{A}$, $A_\tau \in \mathbf{A}_\tau$ and for any $\varepsilon > 0$ one can find a $\delta > 0$, such that for any $t \geq t_0$ and for any initial function $\varphi_{t_0\tau}(\theta)$ given on the segment $[t_0 - \tau; t_0]$ and satisfying the condition $||\varphi_{t_0\tau}||_\tau < \delta$, the solution $x(t, \varphi_{t_0\tau})$ of the system (2.2) satisfies the conditions

$$||x(t, \varphi_{t_0\tau})|| < \varepsilon$$

and

$$\lim_{t\to\infty} ||x(t, \varphi_{t_0\tau})|| = 0,$$

where $||\varphi_{t_0\tau}||_\tau = \max\limits_{t_0-\tau\le\theta\le t_0} ||\varphi_{t_0\tau}(\theta)||$ and $||\cdot||$ is the Euclidean norm in $\mathbb{R}^n$.

In the following we consider conditions under which the interval time-delay system (2.1) will be asymptotically stable in the sense of Definition 2.1.

3. Preliminaries

Nowadays there are two approaches to investigate the asymptotic stability of time-delay control systems given in the state space. Both approaches are based on the idea of using Lyapunov's direct method. The essence of that concerning differential equations of the perturbed motion consists in choosing some continuous function $V(x_1, x_2, \ldots, x_n)$. It plays the role of a general distance from the origin of coordinates. The decrease of the chosen function along motion trajectories of the system under investigation means the asymptotic stability of the equilibrium position $x(t, \varphi_{t_0\tau}) \equiv 0$. Since the direct extension of this method to the class of differential time-delay equations has some limitations consisting, mainly, in severity of finding conditions under which Lyapunov's function decreases along the motion trajectories of the system, in both approaches mentioned above the main effort was done to overcome these difficulties. The first of them is based on Razumikhin's principle and scalar-optimization functions [10]. However, the idea by Krasovsky [9] who proposed to use functionals possessing similar properties instead of Lyapunov's functions appears to be more fruitful. It is remarkable, that the second theorem proved in [9] by Krasovsky on the asymptotic stability can be applied to the class of interval differential time-delay equations (2.1) without any significant changes. This theorem concerning (2.1) will be formulated as follows:

THEOREM 3.1. *Let*

$$||x(t)||_{\tau_2} = \left(\int_{-\tau}^{0} \sum_{i=1}^{n} x_i^2(t+s)\,\mathrm{d}s \right)^{1/2}.$$

The equilibrium position $x(t, \varphi_{t_0\tau}) \equiv 0$ *with the initial function* $\varphi_{t_0\tau}(\theta) \equiv 0$, $t_0 - \tau \le \theta \le t_0$ *of the system (2.1) is asymptotically stable, if there exist a functional* $V(x(t+s), t)$*, some monotonic increasing functions* $W_1(r)$, $W_2(r)$, $r \ge 0$, $W_1(0) = W_2(0) = 0$ *and some continuous positive functions* $W(r)$ *and* $\psi(r)$, $r > 0$, *such that*

i) $$V\big(x(t+s), t\big) \le W_1\big(||x(t)||\big) + W_2\big(||x(t)||_{\tau_2}\big), \tag{3.1}$$

ii) $$V\big(x(t+s), t\big) \ge W\big(||x(t)||\big), \tag{3.2}$$

iii) $$\lim_{\Delta t\to +0} \sup \frac{\Delta V}{\Delta t} \le -\psi\big(||x(t)||\big), \tag{3.3}$$

where the variable s changes within $-\tau \leq s \leq 0$.

For the constant matrices $A \in \mathbf{A}$ and $A_\tau \in \mathbf{A}_\tau$ we can choose the functional in the form $V(x(t+s), t) = V(x(t+s))$, i.e., it does not depend directly on t [9]. It is easy to see, that the functional

$$V\big(x(t+s)\big) = x^T(t)Hx(t) + \int_{-\tau}^{0} x^T(t+v)Dx(t+v)\,\mathrm{d}v, \tag{3.4}$$

where $H = H^T \in \mathbb{R}^{n\times n}$ is a positive definite symmetric matrix, $D = \operatorname{diag}\{d_i > 0,\ 1 \leq i \leq n\}$ is a diagonal matrix, satisfies the conditions (3.1) and (3.2) of the theorem by Krasovsky. However, finding the right upper derivative $\lim\limits_{\Delta t\to +0} \sup(\Delta V / \Delta t)$ for the chosen functional (3.4) by virtue of (2.1), where supremum is taken over all matrices $A \in \mathbf{A}$ and $A_\tau \in \mathbf{A}_\tau$, may appear rather difficult in practice. For thin matrices $A \in \mathbf{A}$ and $A_\tau \in \mathbf{A}_\tau$ the value of the right upper derivative coincides with the usual time derivative of the functional (3.4) by virtue of (2.2).

4. Main result

In this section we shall prove Theorem 4.1 that gives us a sufficient condition for the asymptotic stability of the equilibrium position $x(t, \varphi_{t_0\tau}) \equiv 0$ of the interval time-delay system (2.1). For our further discussions we need the following definitions.

DEFINITION 4.1. An interval square matrix $\mathbf{Q} \in \mathbb{IR}^{n\times n}$, $\mathbf{Q} = (\mathbf{q}_{ij})$, $\mathbf{q}_{ij} = [\underline{\mathbf{q}}_{ij}, \overline{\mathbf{q}}_{ij}]$, $1 \leq i, j \leq n$ is said to be positive definite, if any matrix $Q \in \mathbf{Q}$ is positive definite, i.e., $x^TQx > 0$ for all $Q \in \mathbf{Q}$ and all $x \in \mathbb{R}^n \setminus \{0\}$.

DEFINITION 4.2. [5] Let $\underline{\mathbf{Q}}^{\mathrm{sym}}, \overline{\mathbf{Q}}^{\mathrm{sym}} \in \mathbb{R}^{n\times n}$ be symmetric matrices and let the inequality sign between matrices be understood componentwise. The set

$$\mathbf{Q}^{\mathrm{sym}} = [\underline{\mathbf{Q}}^{\mathrm{sym}}, \overline{\mathbf{Q}}^{\mathrm{sym}}] = \{Q \in \mathbb{R}^{n\times n} \mid Q = Q^T,\ \underline{\mathbf{Q}}^{\mathrm{sym}} \leq Q \leq \overline{\mathbf{Q}}^{\mathrm{sym}}\},$$

is said to be a symmetric interval matrix. It is written as $\mathbf{Q}^{\mathrm{sym}} = (\mathbf{Q}^{\mathrm{sym}})^T$.

It is easy to see from Definition 4.2, that $\mathbf{Q}^{\mathrm{sym}} \notin \mathbb{IR}^{n\times n}$ in the usual sense for $n > 1$ when at least one off-diagonal element is a non-degenerate interval. In [5] such matrices are called *dependent interval matrices*, since the off-diagonal elements lying symmetrically relative to the main diagonal of this matrix depend on each other.

In the present paper the solution of our task is based on the following quite obvious assertion.

By virtue of the equations (2.2) let for arbitrary matrices $A \in \mathbf{A}$ *and* $A_\tau \in \mathbf{A}_\tau$ *the time derivative of the functional (3.4) be negative on the motion trajectories of the system (2.2). Then the relation (3.3) is valid.*

According to the chosen approach let us find the time derivative of the functional (3.4) along the trajectories of the system (2.2) for arbitrary, but fixed matrices $A \in \mathbf{A}$ and $A_\tau \in \mathbf{A}_\tau$:

$$\begin{aligned}\frac{\mathrm{d}V(x(t+s))}{\mathrm{d}t} &= \dot{x}^T(t)Hx(t) + x^T(t)H\dot{x}(t) + x^T(t)Dx(t) - x^T(t-\tau)Dx(t-\tau)\\ &= \big(x^T(t)A^T + x^T(t-\tau)A_\tau^T\big)Hx(t) + x^T(t)H\big(Ax(t) + A_\tau x(t-\tau)\big)\\ &\quad + x^T(t)Dx(t) - x^T(t-\tau)Dx(t-\tau)\\ &= x^T(t)(A^TH + HA + D)x(t) + x^T(t-\tau)A_\tau^T Hx(t)\\ &\quad + x^T(t)HA_\tau x(t-\tau) - x^T(t-\tau)Dx(t-\tau). \qquad (4.1)\end{aligned}$$

Note, that the variable s takes here only the two values 0 and $-\tau$ [9]. Now we introduce the vector $y(t+s) = (y_i(t+s))$, $1 \leq i \leq 2n$,

$$y(t+s) = \begin{pmatrix} x(t) \\ x(t-\tau) \end{pmatrix}$$

and the matrix $C \in \mathbb{R}^{2n\times 2n}$

$$C = \begin{pmatrix} A^TH + HA + D & HA_\tau \\ A_\tau^T H & -D \end{pmatrix}. \qquad (4.2)$$

Then $\mathrm{d}V(x(t+s)) / \mathrm{d}t$ can be rewritten as

$$\frac{\mathrm{d}V(x(t+s))}{\mathrm{d}t} = y^T(t+s)Cy(t+s). \qquad (4.3)$$

Therefore, if the time derivative (4.1) of the functional $V(x(t+s))$ is negative on the motion trajectories of the system (2.2) for any matrices $A \in \mathbf{A}$ and $A_\tau \in \mathbf{A}_\tau$, then the solution $x(t) \equiv 0$ of the system (2.2) is asymptotically stable for any $A \in \mathbf{A}$ and $A_\tau \in \mathbf{A}_\tau$. Hence, our interval time-delay system (2.1) is asymptotically stable in the sense of Definition 2.1.

Let $\mathbf{Q}^{\mathrm{sym}}$ be some positive definite symmetric interval matrix. For the interval matrices $\mathbf{A}$ and $\mathbf{Q}^{\mathrm{sym}}$ we now construct the following set of matrices $H \in \mathbb{R}^{n\times n}$

$$\begin{aligned}\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{Q}^{\mathrm{sym}}) &= \{H \in \mathbb{R}^{n\times n} \mid (\forall A \in \mathbf{A})(\exists Q \in \mathbf{Q}^{\mathrm{sym}})(A^TH + HA = -Q)\}\\ &= \{H \in \mathbb{R}^{n\times n} \mid (\forall A \in \mathbf{A})(A^TH + HA \in -\mathbf{Q}^{\mathrm{sym}})\}. \qquad (4.4)\end{aligned}$$

This set is called a tolerable solution set [13], [14] of Lyapunov's interval matrix equation

$$\mathbf{A}^T H + H\mathbf{A} = -\mathbf{Q}^{\mathrm{sym}}. \qquad (4.5)$$

The following theorem is valid.

THEOREM 4.1. *Let for the given interval matrix* $\mathbf{A}$ *and some positive definite symmetric interval matrix* $\mathbf{Q}^{\mathrm{sym}}$ *the tolerable solution set (4.4) be not empty, i.e.,* $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{Q}^{\mathrm{sym}}) \neq \emptyset$. *Assume that some symmetric matrix* $\tilde{H} = \tilde{H}^T \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{Q}^{\mathrm{sym}})$ *is positive definite, and that there exist constants* $d_i > 0$, $1 \leq i \leq n$, *such that for the given interval matrix* $\mathbf{A}_\tau$ *the interval matrix*

$$\mathbf{C} = \begin{pmatrix} \mathbf{A}^T\tilde{H} + \tilde{H}\mathbf{A} + D & \tilde{H}\mathbf{A}_\tau \\ \mathbf{A}_\tau\tilde{H} & -D \end{pmatrix} \tag{4.6}$$

is negative definite. Then the equilibrium position $x(t, \varphi_{t_0\tau}) \equiv 0$ *of the interval time-delay system (2.1) is asymptotically stable.*

Proof. By assumption for any matrices $A \in \mathbf{A}$ and $A_\tau \in \mathbf{A}_\tau$ the matrix

$$\tilde{C} = \begin{pmatrix} A^T\tilde{H} + \tilde{H}A + D & \tilde{H}A_\tau \\ A_\tau^T\tilde{H} & -D \end{pmatrix}$$

is negative definite. By virtue of (4.3) the time derivative (4.1) of the functional (3.4) is negative on the trajectories of (2.2) for any $A \in \mathbf{A}$ and $A_\tau \in \mathbf{A}_\tau$, and for $\tilde{H}$ instead of H. Thus, according to our assertion formulated above the conditions of Theorem 4.1 are satisfied. Therefore the interval time-delay system (2.1) is asymptotically stable. The theorem is proved. □

Now let us discuss how Theorem 4.1 can be applied in practice. The questions of choosing a positive definite symmetric interval matrix, positive constants $d_i > 0$, $1 \leq i \leq n$ and investigating non-emptiness of the tolerable solution set of Lyapunov's interval matrix equation may appear. Choosing a positive definite symmetric interval matrix is certainly an important task. Here we can only give some small hints of its choice. One of the simplest way to construct a positive definite symmetric interval matrix is to use Gershgorin's circles that are well-known in linear algebra. The same way is also suitable for choosing positive constants $d_i > 0$, $1 \leq i \leq n$. Besides this way one can use the natural interval extension [15] of Silvester's criterion of negative definiteness of a real matrix for choosing positive constants $d_i > 0$, $1 \leq i \leq n$, such that the interval matrix (4.6) is negative definite. Applying this criterion allows to obtain algebraic inequalities concerning d_i, $1 \leq i \leq n$. It is also possible to apply this criterion for checking the positive definiteness of the matrix $\tilde{H}$ (main diagonal minors of the matrix $\tilde{H}$ are positive).

Investigating non-emptiness of the tolerable solution set of interval systems of linear algebraic equations is well studied now by Shary in the works [13], [14]. However, as pointed out above, $\mathbf{Q}^{\mathrm{sym}} \notin \mathbb{IR}^{n\times n}$ for $n > 1$ when at least one off-diagonal element is a non-degenerate interval. Therefore, in order to apply the results in [13], [14] for investigating non-emptiness of (4.5) we need some auxiliary constructions: using the given positive definite symmetric interval matrix $\mathbf{Q}^{\text{sym}}$ we construct the matrix $\mathbf{Q} = [\underline{\mathbf{Q}}^{\text{sym}}; \overline{\mathbf{Q}}^{\text{sym}}] \in \mathbb{IR}^{n\times n}$ and the set

$$\begin{aligned}\Sigma_{\forall\exists}(\mathbf{A},\mathbf{Q}) &= \{H \in \mathbb{R}^{n\times n} \mid (\forall A \in \mathbf{A})(\exists Q \in \mathbf{Q})(A^T H + HA = -Q)\}\\ &= \{H \in \mathbb{R}^{n\times n} \mid (\forall A \in \mathbf{A})(A^T H + HA \in -\mathbf{Q}\}\\ &\supseteq \{H \in \mathbb{R}^{n\times n} \mid \mathbf{A}^T H + H\mathbf{A} \subseteq -\mathbf{Q}\},\end{aligned} \tag{4.7}$$

where $\mathbf{A}^T H + H\mathbf{A}$ is computed using interval arithmetic. The set (4.7) is a tolerable solution set of the interval matrix equation

$$\mathbf{A}^T H + H\mathbf{A} = -\mathbf{Q}, \tag{4.8}$$

having the interval matrix $\mathbf{Q} \in \mathbb{IR}^{n\times n}$ without dependencies in the right hand-side. We shall call the equation (4.8) an auxiliary interval matrix equation of Lyapunov.

THEOREM 4.2. *Let the tolerable solution set (4.7) of the auxiliary interval matrix equation of Lyapunov (4.8) be not empty, and let some symmetric matrix $\tilde{H} = \tilde{H}^T$ belong to this set, i.e.*

$$\tilde{H} = \tilde{H}^T \in \Sigma_{\forall\exists}(\mathbf{A},\mathbf{Q}) \neq \emptyset. \tag{4.9}$$

Then

$$\{\tilde{H} \in \mathbb{R}^{n\times n} \mid \tilde{H} \in \Sigma_{\forall\exists}(\mathbf{A},\mathbf{Q}),\ \tilde{H} = \tilde{H}^T\} \subseteq \Sigma_{\forall\exists}(\mathbf{A},\mathbf{Q}^{\text{sym}}). \tag{4.10}$$

Proof. Since

$$(A^T\tilde{H} + \tilde{H}A)^T = \tilde{H}^T A + A^T\tilde{H}^T = \tilde{H}A + A^T\tilde{H} = A^T\tilde{H} + \tilde{H}A,\ \ \forall A \in \mathbf{A}, \tag{4.11}$$

the matrix $A^T\tilde{H}+\tilde{H}A$ is symmetric for any $A \in \mathbf{A}$. Therefore, there exists a symmetric matrix $\tilde{Q} = \tilde{Q}^T \in \mathbf{Q}$ such that

$$A^T\tilde{H} + \tilde{H}A = -\tilde{Q},$$

whence

$$A^T\tilde{H} + \tilde{H}A \in -\mathbf{Q}^{\text{sym}}.$$

Since the latter is true for any symmetric matrix $\tilde{H}$ satisfying (4.9) the inclusion (4.10) is valid and the theorem is proved. □

In view of (4.7) we consider the following simple algebraic relation concerning the matrix H:

$$|\text{mid}\,\mathbf{A}^T H + H\text{mid}\,\mathbf{A} + \text{mid}\,\mathbf{Q}| \leq \text{rad}\,\mathbf{Q} - \text{rad}\,\mathbf{A}^T|H| - |H|\text{rad}\,\mathbf{A}. \tag{4.12}$$

The relation (4.12) is similar to that offered by Rohn [12] and characterizes a class of matrices H belonging to the set (4.7). From the relation (4.12) one can see, if the matrix $\tilde{H}$ is a solution of the "midpoint" thin matrix equation of Lyapunov

$$(\text{mid}\,\mathbf{A}^T)\tilde{H} + \tilde{H}(\text{mid}\,\mathbf{A}) = -\text{mid}\,\mathbf{Q} \tag{4.13}$$

and satisfies the condition

$$\text{rad}\,\mathbf{A}^T|\tilde{H}| + |\tilde{H}|\text{rad}\,\mathbf{A} \leq \text{rad}\,\mathbf{Q}, \tag{4.14}$$

where the inequality sign is understood componentwise, then

$$\tilde{H} \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{Q}).$$

The matrix equation (4.13) represents a system of linear algebraic equations. To find its solution is possible by well-known methods of linear algebra. If the matrix mid $\mathbf{A}$ is asymptotically stable, then the solution $\tilde{H}$ of the matrix equation (4.13) is always symmetric for the symmetric matrix mid $\mathbf{Q}$. We mentioned that non-emptiness of the tolerable solution set of linear interval systems was also studied in [13], [14].

5. Numeric example

Let us consider an interval time-delay system for $n = 3$

$$\begin{cases} \dot{x}_1(t) = [0; 0.01]x_1(t) + x_2(t) + [-0.01; 0.01]x_2(t-\tau), \\ \dot{x}_2(t) = [-1; -0.9]x_1(t) - x_2(t) + [-0.1; 0]x_3(t) + [0; 0.05]x_1(t-\tau), \\ \dot{x}_3(t) = [0; 0.1]x_2(t) - x_3(t) + [0.1; 0.2]x_2(t-\tau) - [0.2; 0.4]x_3(t-\tau), \end{cases}$$

$$x_i(\theta) = \varphi_{i_{t_0\tau}}(\theta), \quad \theta \in [t_0 - \tau, t_0], \quad 1 \leq i \leq 3.$$

It is easy to see, that

$$\mathbf{A} = \begin{pmatrix} [0; 0.1] & 1 & 0 \\ [-1; -0.9] & -1 & [-0.1; 0] \\ 0 & [0; 0.1] & -1 \end{pmatrix}, \tag{5.1}$$

$$\mathbf{A}_\tau = \begin{pmatrix} 0 & [-0.01; 0.01] & 0 \\ [0; 0.05] & 0 & 0 \\ 0 & [0.1; 0.2] & [-0.4; -0.2] \end{pmatrix}.$$

The positive definite interval matrix

$$\mathbf{Q} = \begin{pmatrix} [\ \ 0.6; 1.4] & [-0.2; 0.2] & [-0.2; 0.2] \\ [-0.2; 0.2] & [\ \ 0.9; 1.1] & [-0.2; 0.2] \\ [-0.2; 0.2] & [-0.2; 0.2] & [\ \ 0.9; 1.1] \end{pmatrix} \tag{5.2}$$

has the unit matrix as its midpoint, i.e.

$$\text{mid}\,\mathbf{Q} = E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The solution of the "midpoint" thin matrix equation of Lyapunov (4.13) for

$$\text{mid}\,\mathbf{A} = \begin{pmatrix} 0.05 & 1 & 0 \\ -0.95 & -1 & -0.05 \\ 0 & 0.05 & -1 \end{pmatrix}$$

and mid $\mathbf{Q} = E$ is:

$$\tilde{H} = \begin{pmatrix} 1.638414 & 0.612548 & -0.011319 \\ 0.612548 & 1.111502 & -0.020921 \\ -0.011319 & -0.020921 & 0.501046 \end{pmatrix}, \tag{5.3}$$

that satisfies the matrix inequality (4.14), since

$$\text{rad}\,\mathbf{A}^T|\tilde{H}| + |\tilde{H}|\text{rad}\,\mathbf{A} = \begin{pmatrix} 0.2250962 & 0.08676845 & 0.0322394 \\ 0.08676845 & 0.0020921 & 0.0806274 \\ 0.0322394 & 0.0806274 & 0.0020921 \end{pmatrix},$$

$$\text{rad}\,\mathbf{Q} = \begin{pmatrix} 0.4 & 0.2 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.1 \end{pmatrix}, \qquad \text{rad}\,\mathbf{A} = \begin{pmatrix} 0.05 & 0 & 0 \\ 0.05 & 0 & 0.05 \\ 0 & 0.05 & 0 \end{pmatrix}.$$

The matrix (5.3) is positive definite, and therefore the interval matrix (5.1) under consideration is asymptotically stable, as in [3]. Let us determine the matrix D from the condition of negative definiteness of the matrix (4.6), so we have

$$\mathbf{A}_\tau^T\tilde{H} = \begin{pmatrix} [\ \ 0.00000000; \ \ 0.03062730] & [\ \ 0.00000000; \ \ 0.05557510] & [-0.00104605; \ \ 0.00000000] \\ [-0.01864794; -0.01751604] & [-0.01030966; -0.00821756] & [\ \ 0.05021779; \ \ 0.10032239] \\ [\ \ 0.00226380; \ \ 0.00452760] & [\ \ 0.00418420; \ \ 0.00836840] & [-0.20041840; -0.10020920] \end{pmatrix},$$

$$\tilde{H}\mathbf{A}_\tau = \begin{pmatrix} [\ \ 0.00000000; 0.03062730] & [-0.01864794; -0.01751604] & [\ \ 0.00226380; \ \ 0.00452760] \\ [\ \ 0.00000000; 0.05557510] & [-0.01030966; -0.00821756] & [\ \ 0.00418420; \ \ 0.00836840] \\ [-0.00104605; 0.00000000] & [\ \ 0.05021779; \ \ 0.10032239] & [-0.20041840; -0.10020920] \end{pmatrix}.$$

Rounding the obtained results to the nearest value having 3 digit after point we write down the interval matrix $\mathbf{C}$ as

$$\mathbf{C} = \begin{pmatrix} [-1.225; -0.775] + d_1 & [-0.087; \ 0.087] & [-0.032; \ 0.032] & [\ 0.000; 0.031] & [-0.019; -0.018] & [\ 0.002; \ 0.005] \\ [-0.087; \ 0.087] & [-1.002; -0.998] + d_2 & [-0.081; \ 0.081] & [\ 0.000; 0.056] & [-0.010; -0.008] & [\ 0.004; \ 0.008] \\ [-0.032; \ 0.032] & [-0.081; \ 0.081] & [-1.002; -0.998] + d_3 & [-0.001; 0.000] & [\ 0.050; \ 0.100] & [-0.200; -0.100] \\ [\ 0.000; \ 0.031] & [\ 0.000; \ 0.056] & [-0.001; \ 0.000] & -d_1 & 0 & 0 \\ [-0.019; -0.018] & [-0.010; -0.008] & [\ 0.050; \ 0.100] & 0 & -d_2 & 0 \\ [\ 0.002; \ 0.005] & [\ 0.004; \ 0.008] & [-0.200; -0.100] & 0 & 0 & -d_3 \end{pmatrix},$$

where $\underline{\mathbf{C}} = \underline{\mathbf{C}}^T$, $\overline{\mathbf{C}} = \overline{\mathbf{C}}^T$. It is easy to see, that the choice of $d_1 = 0.1 > 0$, $d_2 = 0.2 > 0$, and $d_3 = 0.3 > 0$ makes the interval matrix $\mathbf{C}$ negative definite, because Gershgorin's circles are in the left half-plane of the complex plane. The last mentioned means, that the interval time-delay system (2.1) is asymptotically stable.

6. Conclusion

Applying Lyapunov–Krasovsky functional for investigating the asymptotic stability of interval time-delay system allows to obtain a sufficient condition of the asymptotic stability of the indicated class of systems.

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