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Linear Interval Tolerance Problem and Linear Programming Techniques

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Abstract. In this paper, we consider the linear interval tolerance problem, which consists of finding the largest interval vector included in $\Sigma_{\forall,\exists}([A],[b]) = \{x \in \mathbb{R}^n \mid \forall A \in [A],\ \exists b \in [b],\ Ax = b\}$. We describe two different polyhedrons that represent subsets of all possible interval vectors in $\Sigma_{\forall,\exists}([A],[b])$, and we provide a new definition of the optimality of an interval vector included in $\Sigma_{\forall,\exists}([A],[b])$. Finally, we show how the Simplex algorithm can be applied to find an optimal interval vector in $\Sigma_{\forall,\exists}([A],[b])$.

1. Introduction

In this paper we propose two algorithms for handling the interval tolerance problem [4], [7]–[9], [11], [12] based on the Simplex algorithm [2], [13] used in linear programming. The approach for combining interval linear systems and Simplex algorithm is described in [1], [3].

The set $\Sigma_{\forall,\exists}([A],[b])$ is defined by

$$\Sigma_{\forall,\exists}([A],[b]) = \{x \in \mathbb{R}^n \mid \forall A \in [A],\ \exists b \in [b],\ Ax = b\}$$

or, equivalently by

$$\Sigma_{\forall,\exists}([A],[b]) = \{x \in \mathbb{R}^n \mid [A]x \subseteq [b]\},$$

the product being evaluated with interval arithmetic.

Shary [12] gives a very interesting and practical interpretation of this set. Suppose that the output y of a physical system is given by $y = Ax$, where x is the input of the system and A is any of the possible matrices of $[A]$. Then, for any input satisfying

$$x \in [x] \subseteq \Sigma_{\forall,\exists}([A],[b]),$$

we can ensure that the output Ax will belong to the tolerable set of outputs $[b]$, whatever the values of A in $[A]$ are. Following Shary [12], we use the notation $\Sigma_{\forall,\exists}([A],[b])$ to exhibit its similarity with the united solution set

$$\Sigma_{\exists,\exists}([A],[b]) = \{x \in \mathbb{R}^n \mid \exists A \in [A],\ \exists b \in [b],\ Ax = b\}$$

and the controllable solution set

$$\Sigma_{\exists,\forall}([A],[b]) = \{x \in \mathbb{R}^n \mid \forall b \in [b],\ \exists A \in [A],\ Ax = b\}.$$

Since we need the output to belong to $[b]$ for any choice of $A \in [A]$, our aim is to find a practical subset of the set of all possible interval vectors in $\Sigma_{\forall,\exists}([A],[b])$.

In Sections 3 and 4, we describe two different polyhedrons which define such subsets. Finally, we show (Section 5) how to define the "optimality" of an interval vector included in $\Sigma_{\forall,\exists}([A],[b])$, and how to apply the Simplex algorithm to compute this "optimal" interval vector.

2. Point Matrix by Interval Vector Product

We will first prove a characterization of the condition

$$\forall x \in [x], \quad Ax \in [b].$$

It is clear that no overestimation occurs when performing the product of an interval matrix by a point vector, since each component of the matrix is used only once. This property does not hold when considering the product of a point matrix by an interval vector (see [10], for instance). Nevertheless, the following property holds.

LEMMA 2.1. *The following assertions are equivalent:*

P1: $\forall x \in [x],\ Ax \in [b]$,

P2: $A[x] \subseteq [b]$*, the product being evaluated with interval arithmetic.*

Proof. Assertion P2 obviously implies Assertion P1. Let us now consider $[y] = A[x]$. By expanding the previous equality component-wise, it is clear that the bounds of each component of $[y]$ can be reached for a good choice of $x \in [x]$. Therefore P1 implies P2. □

Thus, it is of interest to obtain a practical characterization of $A[x]$.

Let A^+ and A^- be the matrices corresponding respectively to the positive and the negative parts of the matrix A; they are characterized by

$$A^+ \geq 0, \quad A^- \geq 0, \quad A = A^+ - A^-, \quad |A| = A^+ + A^-.$$

Let $[x] = [\underline{x}, \overline{x}]$, $\underline{x} \leq \overline{x}$ (all the inequalities between vectors or matrices must be understood component-wise).

It is easy to prove that

$$A[x] = [-A^-\overline{x} + A^+\underline{x},\ A^+\overline{x} - A^-\underline{x}].$$

Therefore,

$$(\forall x \in [x],\ Ax \in [b]) \iff \begin{cases} A^+\underline{x} - A^-\overline{x} \geq \underline{b}, \\ A^+\overline{x} - A^-\underline{x} \leq \overline{b}. \end{cases}$$

By expressing $[x]$ with the midpoint-radius format, the last equivalence can be translated into the following form:

LEMMA 2.2. *If we set*

$$x_m = \frac{\underline{x} + \overline{x}}{2} \quad \text{and} \quad x_d = \frac{\overline{x} - \underline{x}}{2},$$

then

$$(\forall x \in [x],\ Ax \in [b]) \Leftrightarrow \begin{cases} Ax_m - |A|x_d & \geq \underline{b}, \\ Ax_m + |A|x_d & \leq \overline{b}. \end{cases}$$

Let us define

$$\mathcal{S} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{2n} \mid x_1 \leq x_2 \text{ and } [x_1, x_2] \subseteq \Sigma_{\forall,\exists}([A], [b]) \right\}$$

which describes the set of all possible interval vectors included in $\Sigma_{\forall,\exists}([A], [b])$. In the next section, we use the characterization of the condition

$$(\forall x \in [x],\ Ax \in [b])$$

obtained above to describe a subset of $\mathcal{S}$.

3. First Characterization of a Subset of $\mathcal{S}$

3.1. PRACTICAL CHARACTERIZATION OF $\Sigma_{\forall,\exists}([A], [b])$

Rohn [7] has proved the following theorem:

THEOREM 3.1. *For any vector* $x \in \mathbb{R}^n$, $x \in \Sigma_{\forall,\exists}([A], [b])$ *if and only if there exists* $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$ *such that* $x = x_1 - x_2$ *and* (x_1, x_2) *satisfies the following system of linear inequalities:*

$$\begin{pmatrix} B \\ -I_{2n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} b' \\ 0 \end{pmatrix},$$

where

$$B = \begin{pmatrix} A_c + \Delta A & -(A_c - \Delta A) \\ -(A_c - \Delta A) & (A_c + \Delta A) \end{pmatrix} \quad \text{and} \quad b' = \begin{pmatrix} \overline{b} \\ -\underline{b} \end{pmatrix},$$

with $[A] = [A_c - \Delta A,\ A_c + \Delta A]$.

This theorem gives a practical description of the solution set, which does not involve interval sets. In the next section, we use this characterization to obtain a subset of $\mathcal{S}$.

3.2. How to Obtain an Interval Vector in $\Sigma_{\forall,\exists}([A],[b])$ and Containing a Fixed Point x_0

In this section, we construct a subset of the set of the interval vectors $[x]$ satisfying the following condition

$$x_0 \in [x] \subseteq \Sigma_{\forall,\exists}([A],[b]),$$

where x_0 is an a priori known element of $\Sigma_{\forall,\exists}([A],[b])$. In what follows, for any vector $x \in \mathbb{R}^n$, we denote by x^+ and x^- the vectors defined by:

$$\begin{cases} x^+ = \dfrac{|x|+x}{2}, \\ x^- = \dfrac{|x|-x}{2}. \end{cases}$$

THEOREM 3.2. *Notations of Theorem 3.1 are still valid.*
Let

$$x_0 \in \Sigma_{\forall,\exists}([A],[b]), \quad b'' = b' - B\begin{pmatrix} x_0^+ \\ x_0^- \end{pmatrix},$$

and

$$\mathcal{S}_1 = \left\{ X \mid X = \begin{pmatrix} x_0 - 2x_2 \\ x_0 + 2x_1 \end{pmatrix} \text{ such that } \begin{pmatrix} B + |B| \\ -I_{2n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} b'' \\ 0 \end{pmatrix} \right\}.$$

Then,

$$\mathcal{S}_1 \subseteq \mathcal{S}.$$

Proof. Let us suppose that $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{2n}$ satisfies the following condition

$$\begin{pmatrix} B + |B| \\ -I_{2n} \end{pmatrix} X \leq \begin{pmatrix} b'' \\ 0 \end{pmatrix}.$$

We can easily check that

$$\begin{pmatrix} B & |B| \\ 0 & -I_{2n} \end{pmatrix} \begin{pmatrix} X + \begin{pmatrix} x_0^+ \\ x_0^- \end{pmatrix} \\ X \end{pmatrix} \leq \begin{pmatrix} b' \\ 0 \end{pmatrix}.$$

Moreover,

$$-I_{2n}\left(X + \begin{pmatrix} x_0^+ \\ x_0^- \end{pmatrix} \right) + I_{2n}X = -\begin{pmatrix} x_0^+ \\ x_0^- \end{pmatrix} \leq 0$$

and therefore,

$$\begin{pmatrix} B & |B| \\ -I_{2n} & I_{2n} \\ 0 & -I_{2n} \end{pmatrix} \begin{pmatrix} X + \begin{pmatrix} x_0^+ \\ x_0^- \end{pmatrix} \\ X \end{pmatrix} \leq \begin{pmatrix} b' \\ 0 \\ 0 \end{pmatrix}.$$

If we set

$$\begin{cases} Z^{(m)} = X + \begin{pmatrix} x_0^+ \\ x_0^- \end{pmatrix}, \\ Z^{(d)} = X, \end{cases}$$

the set of inequalities defined above becomes

$$\begin{cases} BZ^{(m)} + |B|Z^{(d)} & \leq & b', \\ Z^{(m)} - Z^{(d)} & \geq & 0, \\ Z^{(d)} & \geq & 0. \end{cases}$$

The last two conditions prove that $[Z] = [Z^{(m)} - Z^{(d)}, Z^{(m)} + Z^{(d)}]$ is a non-negative interval vector. Owing to Lemma 2.2,

$$BZ^{(m)} + |B|Z^{(d)} \leq b' \quad \Leftrightarrow \quad \forall z \in [Z], \;\; Bz \leq b'.$$

Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in [Z]$, then $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ satisfies the conditions of Theorem 3.1, since

$$\begin{cases} B\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} & \leq & b', \\ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} & \geq & 0 \end{cases}$$

and therefore $z_1 - z_2$ belongs to $\Sigma_{\forall,\exists}([A], [b])$.

When $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ describes $[Z]$, $(z_1 - z_2)$ describes

$$[x_0^+ - x_0^- - 2x_2, \, x_0^+ - x_0^- + 2x_1]$$

and therefore,

$$[x_0 - 2x_2, \, x_0 + 2x_1] \subseteq \Sigma_{\forall,\exists}([A], [b]),$$

which achieves the proof of Theorem 3.2. □

4. Second Characterization of a Subset of $\mathcal{S}$

In this section, we present a second approach for dealing with the interval tolerance problem. Rohn [7] has given a characterization of the points belonging to $\Sigma_{\forall,\exists}([A], [b])$ analogous to the Oettli-Prager Theorem [6] for the united solution set $\Sigma_{\exists,\exists}$.

If we set

$$[A] = [A_c - \Delta A, \, A_c + \Delta A]$$

and

$$[b] = [b_c - \Delta b,\ b_c + \Delta b] = [\underline{b}, \overline{b}],$$

then

$$x \in \Sigma_{\forall,\exists}([A],[b]) \iff |A_c\, x - b_c| \le \Delta b - \Delta A|x|.$$

This formula is of practical interest because it gives a characterization of the solution set that does not involve interval sets. It is however hard to use because of the absolute values. In order to get rid of absolute values, we can use the following Theorem [1].

THEOREM 4.1. *If $\underline{x} \le \overline{x}$ and if we set*

$$\begin{cases} \alpha_j = \dfrac{|\overline{x}_j| - |\underline{x}_j|}{\overline{x}_j - \underline{x}_j} & \textit{and} \quad \beta_j = \dfrac{\overline{x}_j|\underline{x}_j| - \underline{x}_j|\overline{x}_j|}{\overline{x}_j - \underline{x}_j} \quad \textit{if} \quad \overline{x}_j \ne \underline{x}_j, \\ \alpha_j = 0 & \textit{and} \quad \beta_j = \overline{x}_j \quad \textit{if} \quad \overline{x}_j = \underline{x}_j, \end{cases}$$

where x_j denotes the j-th component of x, then

$$\forall x \in [\underline{x}, \overline{x}],\ \ \forall j,\ \ 1 \le j \le n,\ \ |x_j| \le \alpha_j x_j + \beta_j.$$

Moreover, if we denote by β the vector whose entries are the β_j's and by D_α the diagonal matrix whose diagonal elements are the α_j's, then

$$x \in [\underline{x}, \overline{x}] \ \Rightarrow\ |x| \le D_\alpha x + \beta.$$

Therefore, if we know an enclosure of

$$\Sigma_{\forall,\exists}([A],[b]) \subseteq [\underline{x}, \overline{x}],$$

(which might be obtained for instance by computing a superset of $\square\Sigma_{\exists,\exists}([A],[b])$ [1], [5], [10]), we can obtain the following characterization of the points of $\Sigma_{\forall,\exists}([A],[b])$.

THEOREM 4.2. *If*

$$\Sigma_{\forall,\exists}([A],[b]) \subseteq [\underline{x}, \overline{x}],$$

then

$$Cx \le c \ \Rightarrow\ x \in \Sigma_{\forall,\exists}([A],[b]),$$

where

$$C = \begin{pmatrix} A_c + \Delta A D_\alpha \\ -A_c + \Delta A D_\alpha \end{pmatrix} \quad \textit{and} \quad c = \begin{pmatrix} b_c + \Delta b - \Delta A \beta \\ -b_c + \Delta b - \Delta A \beta \end{pmatrix}.$$

Proof. Let $x \in [\underline{x}, \overline{x}]$ satisfying

$$Cx \leq c,$$

then

$$|A_c x - b_c| \leq \Delta b - \Delta A(D_\alpha x + \beta).$$

Moreover, since $x \in [\underline{x}, \overline{x}]$, because of Theorem 4.1,

$$-\Delta A(D_\alpha x + \beta) \leq -\Delta A|x|$$

and therefore

$$|A_c x - b| \leq \Delta b - \Delta A|x|. \qquad \square$$

Since we know a characterization of a subset of $\Sigma_{\forall,\exists}([A],[b])$ by means of the inequality

$$Cx \leq c,$$

we can use Lemma 2.2 in order to prove the following theorem.

THEOREM 4.3. *If we set*

$$\mathcal{S}_2 = \left\{ y \;\middle|\; y = \begin{pmatrix} x^{(m)} - x^{(d)} \\ x^{(m)} + x^{(d)} \end{pmatrix}, \; \begin{pmatrix} C & |C| \\ 0 & -I_n \end{pmatrix} \begin{pmatrix} x^{(m)} \\ x^{(d)} \end{pmatrix} \leq \begin{pmatrix} c \\ 0 \end{pmatrix} \right\},$$

where

$$C = \begin{pmatrix} A_c + \Delta A D_\alpha \\ -A_c + \Delta A D_\alpha \end{pmatrix} \quad \textit{and} \quad c = \begin{pmatrix} b_c + \Delta b - \Delta A\beta \\ -b_c + \Delta b - \Delta A\beta \end{pmatrix},$$

then

$$\mathcal{S}_2 \subseteq \mathcal{S}.$$

In the case $\Delta A = 0$, we can notice an analogy between the formulation of Theorem 4.3 and the results obtained by Shary [11], [12]. In our case, we enforce the interval vector $[x^{(m)} - x^{(d)}, x^{(m)} + x^{(d)}]$ to be a classical interval vector by considering the extra condition $x^{(d)} \geq 0$.

5. Choice of the Optimization Problem

In Sections 3 and 4, we have obtained two characterizations ($\mathcal{S}_1$ and $\mathcal{S}_2$) of subsets of the set of all the interval vectors included in $\Sigma_{\forall,\exists}([A],[b])$. These characterizations are obtained by means of convex polyhedrons of $\mathbb{R}^{2n}$. An important issue is to define the "optimality" of an interval vector in $\Sigma_{\forall,\exists}([A],[b])$. Since we want to apply general techniques of linear programming, we need to find a linear objective function. Usually, the optimality of the interval vector is defined with respect to

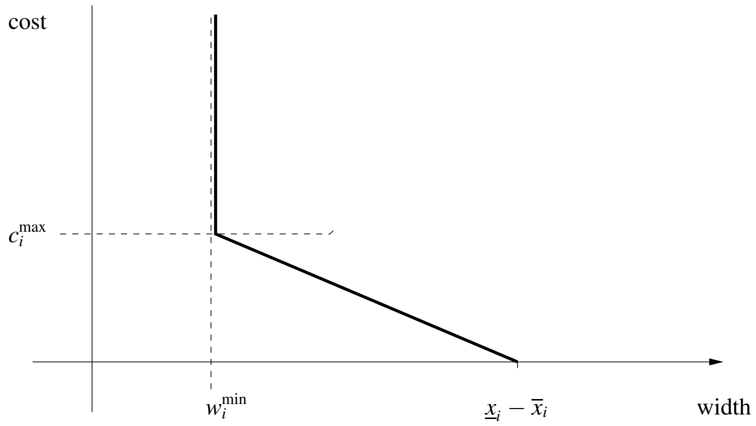


Figure 1. Representation of the cost associated to x_i.

its volume, what is impossible in our context. Nevertheless, we show in the next section how to define a linear objective function which is meaningful with respect to practical interpretation given by Shary [12].

5.1. Cost of an Interval Vector

Usually, solving the interval linear tolerance problem consists of finding an interval vector $[\underline{x}, \overline{x}]$, included in $\Sigma_{\forall,\exists}([A],[b])$ and maximizing $\prod_1^n(\overline{x}_i - \underline{x}_i)$ [5].

Nevertheless, as Shary [12] notices it, solving the interval tolerance problem consists of finding an interval vector $[\underline{x}, \overline{x}]$ such that for any value of the parameters of $[A]$, we altogether get the output response within the required tolerance $[b]$.

Therefore, we can associate a cost to each variable x_i. Intuitively, the narrower the input interval $[\underline{x}_i, \overline{x}_i]$, the more important the cost for ensuring it.

Besides, one can even imagine that for some physical reasons (measure uncertainties, previous computations) it is impossible to have an input interval $[\underline{x}_i, \overline{x}_i]$ narrower than a given threshold $w_i^{\min}$.

The cost associated to the input variable x_i may therefore be defined as in Figure 1.

5.2. Corresponding Constraints and Optimization Function

In this section, we show how the constraints of the simplex and the objective function can be reformulated accordingly to the cost associated to each variable.

The condition

$$\overline{x}_i - \underline{x}_i \geq w_i^{\min}$$

corresponds to changing the constraints

$$x_i^{(d)} \geq 0$$

into

$$x_i^{(d)} \geq \frac{1}{2} w_i^{\min}$$

and the minimization of the global cost is expressed by the minimization of the following objective function

$$\sum_{i=1}^{n} f_i x_i^{(d)}$$

where

$$f_i = -\left(\frac{c_i^{\max}}{\overline{x}_i - \underline{x}_i - w_i^{\min}} \right).$$

6. Sketch of the Algorithms

In Sections 3 and 4, we have defined $\mathcal{S}_1$ and $\mathcal{S}_2$, which describe subsets of the set of all possible interval vectors included in $\Sigma_{\forall,\exists}([A],[b])$. Since both $\mathcal{S}_1$ and $\mathcal{S}_2$ are convex polyhedrons, it is possible to use linear programming for computing the optimal vector (with respect to the objective function defined in Section 5) included in $\mathcal{S}_1$ or $\mathcal{S}_2$ and therefore in $\Sigma_{\forall,\exists}([A],[b])$.

6.1. MINIMIZATION OVER $\mathcal{S}_1$

In this section, we present the sketch of the algorithm to compute the optimal interval vector included in $\mathcal{S}_1$ with respect to the objective function defined in Section 5.

Algorithm 1 requires the resolution of a linear program with $5n$ constraints and $2n$ variables.

6.2. MINIMIZATION OVER $\mathcal{S}_2$

In this section, we present the sketch of the algorithm to compute the optimal interval vector included in $\mathcal{S}_2$ with respect to the objective function defined in Section 5.

Algorithm 2 only requires the resolution of a linear program with $3n$ constraints and $2n$ variables.

7. Numerical Results

7.1. A FEW EXAMPLES WITH $n = 2$

We present numerical results for the two algorithms we have introduced. For each case, Sol_1 represents the enclosure obtained with Algorithm 1, and Sol_2 the one obtained with Algorithm 2.

Algorithm 1. Minimization over $\mathcal{S}_1$.

Solve $A_c x_0 = b_c$.

$$x_0^+ := \frac{|x_0| + x_0}{2},$$

$$x_0^- := \frac{|x_0| - x_0}{2},$$

$$B_1 := \begin{pmatrix} A_c + \Delta A & -(A_c - \Delta A) \\ -(A_c - \Delta A) & (A_c + \Delta A) \end{pmatrix},$$

$$b_1 := \begin{pmatrix} \overline{b} \\ -\underline{b} \end{pmatrix} - B_1 \begin{pmatrix} x_0^+ \\ x_0^- \end{pmatrix},$$

$$C_1 := \begin{pmatrix} B_1 + |B_1| \\ -I_{2n} \\ -2I_n \quad -2I_n \end{pmatrix},$$

$$c_1 := \begin{pmatrix} b_1 \\ 0_{2n} \\ -w^{\min} \end{pmatrix}.$$

Minimize $\left\{ \sum_{i=1}^{n} f_i \left(y_i^{(1)} + y_i^{(2)} \right), \quad C_1 \begin{pmatrix} y^{(1)} \\ y^{(2)} \end{pmatrix} \leq c_1 \right\}.$

Algorithm 2. Minimization over $\mathcal{S}_2$.

$$\alpha_j := \frac{|\overline{x}_j| - |\underline{x}_j|}{\overline{x}_j - \underline{x}_j},$$

$$\beta_j := \frac{\overline{x}_j|\underline{x}_j| - \underline{x}_j|\overline{x}_j|}{\overline{x}_j - \underline{x}_j},$$

$$B_2 := \begin{pmatrix} A_c + \Delta A D_\alpha \\ -A_c + \Delta A D_\alpha \end{pmatrix},$$

$$b_2 := \begin{pmatrix} b_c + \Delta b - \Delta A \beta \\ -b_c + \Delta b - \Delta A \beta \end{pmatrix},$$

$$C_2 := \begin{pmatrix} C & |C| \\ 0 & -I_n \end{pmatrix},$$

$$c_2 := \begin{pmatrix} b_2 \\ -w^{\min} \end{pmatrix}.$$

Minimize $\left\{ \sum_{i=1}^{n} c_i \left(y_i^{(2)} - y_i^{(1)} \right), \quad C_2 \begin{pmatrix} y_i^{(1)} \\ y_i^{(2)} \end{pmatrix} \leq c_2 \right\}.$

The figures show the exact solution set. The exact solution set is obtained with the characterization given by Rohn [7]:

$$x \in \Sigma_{\forall,\exists}([A],[b]) \Longleftrightarrow |A_c\, x - b_c| \leq \Delta b - \Delta A|x|.$$

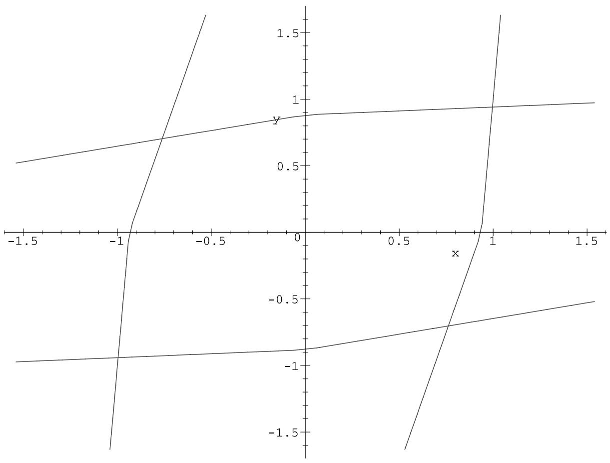


Figure 2.

This formula is of practical interest for computing the exact solution since within each orthant, it is possible to get rid of the absolute value $|x|$. Therefore, within each orthant, we obtain the characterization of the points belonging to $\Sigma_{\forall,\exists}([A],[b])$ by means of a convex polyhedron. Of course, this approach leads to a very slow algorithm, since the number of orthants may be exponential in the size of the matrix (2^n).

The scale of the figures represents the starting enclosure (used to compute Sol_2).

7.1.1. *The Solution Set Is Centered with Respect to 0 (Figure 2)*

$$A_c = \begin{pmatrix} 3 & -0.5 \\ -0.5 & 3 \end{pmatrix}, \quad \Delta A = \begin{pmatrix} 0.2 & 0.3 \\ 0.3 & 0.4 \end{pmatrix}, \quad b_c = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \Delta b = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

In this case, we obtain

$$\mathrm{Sol}_1 = \begin{pmatrix} -0.76 & 0.76 \\ -0.70 & 0.70 \end{pmatrix} \quad \text{and} \quad \mathrm{Sol}_2 = \begin{pmatrix} -0.64 & 0.64 \\ -0.52 & 0.52 \end{pmatrix}.$$

For this example, the results obtained with the first algorithm are better since the solution set is centered with respect to 0, i.e. the worst situation for Theorem 4.1.

7.1.2. *The Initial Enclosure Does Not Intersect Any Axis (Figure 3)*

$$A_c = \begin{pmatrix} 3 & -0.5 \\ -0.5 & 3 \end{pmatrix}, \quad \Delta A = \begin{pmatrix} 0.2 & 0.3 \\ 0.3 & 0.4 \end{pmatrix}, \quad b_c = \begin{pmatrix} 7 \\ 7 \end{pmatrix} \quad \text{and} \quad \Delta b = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

For this problem, we obtain

$$\mathrm{Sol}_1 = \begin{pmatrix} 2.35 & 3.25 \\ 2.60 & 3 \end{pmatrix} \quad \text{and} \quad \mathrm{Sol}_2 = \begin{pmatrix} 2.30 & 3.28 \\ 2.54 & 3.07 \end{pmatrix}.$$

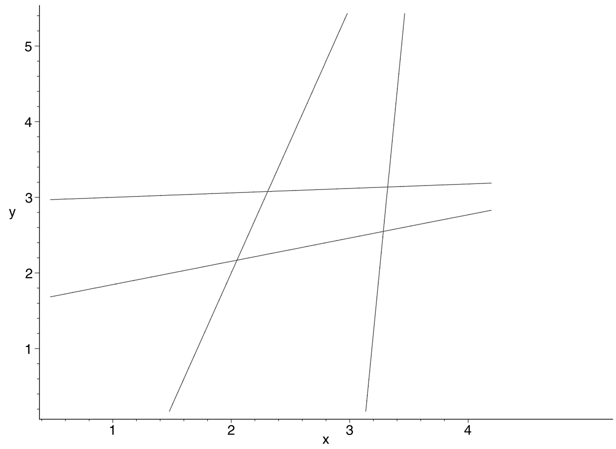


Figure 3.

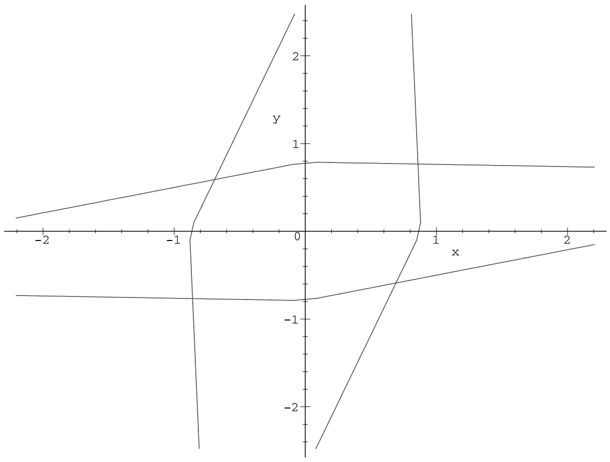


Figure 4.

In this case, the initial enclosure of the solution set does not intersect any axis, and the second algorithm enables us, to compute an optimal solution set.

7.1.3. *Starting from a Large Initial Enclosure (Figure 4)*

$$A_c = \begin{pmatrix} 3 & -0.5 \\ -0.5 & 3 \end{pmatrix}, \quad \Delta A = 2\begin{pmatrix} 0.2 & 0.3 \\ 0.3 & 0.4 \end{pmatrix}, \quad b_c = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \Delta b = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

In this case, we obtain

$$\mathrm{Sol}_1 = \begin{pmatrix} -0.68 & 0.68 \\ -0.58 & 0.58 \end{pmatrix} \quad \text{and} \quad \mathrm{Sol}_2 = \emptyset.$$

The initial enclosure of the solution set is very large, and the second algorithm does not succeed in finding an interval vector included in $\Sigma_{\forall,\exists}([A],[b])$. This suggests the use of Algorithm 2 as an iterative algorithm: if the solution set obtained is not big enough, then one can start with a smaller enclosure. In any case, if the solution set obtained is included into the starting enclosure, then it is acceptable.

7.2. Numerical Results for Higher Dimensions

We plot the evolution of τ versus the matrix order n, where

$$\tau = \frac{\text{perimeter of the interval vector obtained with Algorithm 1}}{\text{perimeter of the interval vector obtained with Algorithm 2}}.$$

The results are displayed for two different situations: In the first situation, b_c, the center of the right hand side, is 0. Therefore, the solution set is usually centered with respect to zero, what leads to a poor enclosure of the absolute value in Algorithm 2.

In the second situation b_c is chosen so that the solution of $A_c x_0 = b_c$ is far from zero. Therefore, the initial enclosure does not intersect any axis, what leads to an exact enclosure of the absolute value in Algorithm 2. In this case, the solution obtained with Algorithm 2 is optimal with respect to the objective function we have defined in Section 5.

For the tests presented in Figure 5, we have $\|A\| = 1$, $\|\Delta A\| = 10^{-4}$, $\|b\| = 1$, and $\|\Delta b\| = 10^{-2}$. It is clear that the first algorithm gives much better results in the first situation ($\tau \in [0.65, 1]$), and only slightly poorer results ($\tau \in [1, 1.0045]$) in the second one for which the second algorithm is optimal.

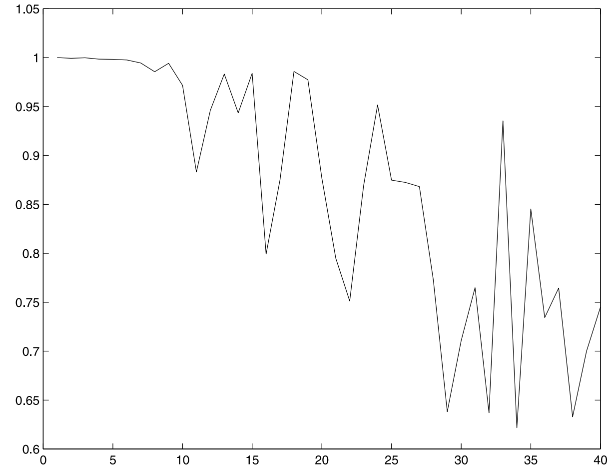
8. Conclusion

The two algorithms that we have proposed enable us to solve the Interval Tolerance Problem. They are both based on the definition of a simplex in spaces of higher dimensions. The linear objective function which has been defined for the criterion of optimality for the sought interval appears to give an interesting definition for the cost of a solution set. Moreover, it is of an easy use since it allows linear programming.

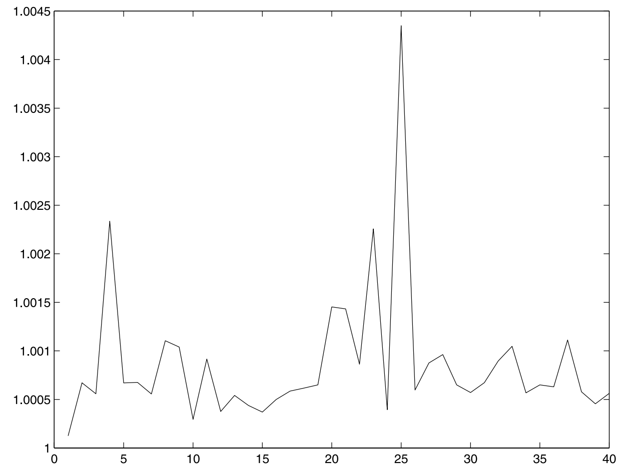
The results obtained are valuable, especially for the first approach. The interest of the second approach is to guaranty an optimal solution when the solution set does not intersect any axis.

References

1. Beaumont, O.: Solving Interval Linear Systems with Linear Programming Techniques, *Linear Algebra and Its Applications* **281** (1998), pp. 293–309.
2. Chvatal, A.: *Linear Programming*, W. H. Freeman and Company, 1983.
3. Jansson, C.: Calculation of Exact Bounds for the Solution Set of Linear Interval Systems, *Linear Algebra and Its Applications* **251** (1997), pp. 321–340.



First situation.



Second situation.

Figure 5. Evaluation of τ with n for two different situations.

4. Kelling, B.: Methods of Solution of Linear Tolerance Problems with Interval Arithmetic, in: *Computer Arithmetic and Enclosure Methods*, North Holland, 1992.
5. Neumaier, A.: *Interval Methods for Systems of Equations*, Cambridge Univ. Press, Cambridge, 1990.
6. Oettli, W. and Prager, W.: Computability of Approximate Solution of Linear Equations with Given Error Bounds for Coefficients and Right-Hand Sides, *Numer. Math.* **6** (1964), pp. 405–409.
7. Rohn, J.: Inner Solutions of Linear Interval Systems, in: Nickel, K. (ed.), *Interval Mathematics 1985*, *Lecture Notes in Computer Science* **212**, Springer, Berlin, 1986, pp. 157–158.
8. Rohn, J.: NP-Hardness Results for Linear Algebraic Problems with Interval Data, in: Herzberger, J. (ed.), *Topics in Validated Computations*, Elsevier Science, 1994.
9. Rohn, J.: Systems of Linear Interval Equations, *Linear Algebra and Its Applications* **126** (1989), pp. 39–78.

10. Rump, S. M.: Verification Methods for Dense and Sparse Systems of Equations, in: Herzberger, J. (ed.), *Topics in Validated Computations*, Elsevier Science, 1994.
11. Shary, S. P.: Algebraic Approach to the Interval Linear Static Identification, Tolerance, and Control Problems, or One More Application of Kaucher Arithmetic, *Reliable Computing* **2** (1) (1996), pp. 3–33.
12. Shary, S. P.: Solving the Linear Interval Tolerance Problem, *Mathematics and Computers in Simulation* **39** (1995), pp. 53–85.
13. Schrijver, A.: *Theory of Linear and Integer Programming*, John Wiley & Sons, 1986.