NORTH-HOLLAND

Controllable Solution Set to Interval Static Systems

Sergey P. Shary

Institute of Computational Technologies
Novosibirsk 630090, Russia

ABSTRACT

For interval systems of algebraic equations, the concept of a *controllable solution set* is introduced and its practical interpretation is given. The main results of the work concern interval linear systems and afford simple sufficient criteria for the controllable solution set to be nonempty and for the intersection of the controllable solution set with some orthants to be empty. © Elsevier Science Inc., 1997

It is customary that the control theory is applied to dynamical systems, either time-continuous or time-discrete. However, closer inspection of its basic notions show that dependence upon the time variable is of secondary importance in the definition of controllability. That is especially pronounced in the abstract mathematical statements of the control problems, where phase trajectories, phase constraints, control subjection, etc. are elements of functional spaces. In the most general form, the logical structure of the notion of controllability is caught by the following scheme.

Let the function $\Phi(c)$ describe a final result of the system's operation depending on the control c. Then the system is (completely) controllable if and only if the predicate

$$(\forall \text{ final state } R \text{ from a marked set})$$

$$(\exists \text{ control subjection } C \text{ from a feasible area})$$

$$(R = \Phi(C)) \tag{1}$$

is true. But in such a form the controllability notion is equally applicable to static systems as well, in which the time variable and the time interval do

APPLIED MATHEMATICS AND COMPUTATION 86:185–196 (1997)

0096-3003/97/$17.00
655 Avenue of the Americas, New York, NY 10010
PII S0096-3003(96)00181-6

not figure at all. Below, we shall render this vague reasoning concrete assuming all data sets involved to be intervals (or interval vectors).

We start with the linear case for simplicity. Let the interval system of linear algebraic equations

$$\mathbf{A}\, x = \mathbf{b} \tag{2}$$

be given with an interval $m \times n$-matrix $\mathbf{A}$ and an interval right-hand side m-vector $\mathbf{b}$. It is common knowledge that (2) is only a formal symbol, which in itself can mean, for instance, a collection of all point linear algebraic systems with the elements that belong to $\mathbf{A}$ and $\mathbf{b}$, respectively. To pose the problem correctly, we should at least define what is meant by the solution or the solution set to (2). So far, the main investigation in the interval analysis have been concerned with the *united solution set*

$$\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})(\, Ax = b)\},$$

formed by solutions of all point systems $Ax = b$ with $A \in \mathbf{A}$ and $b \in \mathbf{b}$ (see [1–3] and the extensive references there) and the *tolerable solution set*

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\exists b \in \mathbf{b})(\, Ax = b)\}$$

formed by all point vectors x such that the product Ax falls into $\mathbf{b}$ for any $A \in \mathbf{A}$ (see [2, 4–6]) (Neumaier [2, 4] and some others call it *restricted solution set* and denote $\Sigma_0(\mathbf{A}, \mathbf{b})$).

But we would like to draw attention to a new solution set of interval linear algebraic systems, namely the set

$$\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall b \in \mathbf{b})(\exists A \in \mathbf{A})(\, Ax = b)\}, \tag{3}$$

or, equivalently, the set

$$\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid \mathbf{A} \cdot x \supseteq \mathbf{b}\},$$

where "$\cdot$" is the common interval matrix multiplication. This set is formed by all point vectors $x \in \mathbb{R}^n$, such that for *any* desired $b \in \mathbf{b}$ we can find the corresponding $A \in \mathbf{A}$ satisfying $Ax = b$. Notice that

$$\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b}) \subseteq \Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})$$

and if $\mathbf{A}$ contains only nonsingular point matrices, then $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ is bounded coincidentally with $\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})$ (as well as connected). What is the content interpretation of the solution set (3)?

Let a static system be given with the structural scheme depicted in Figure 1 and the input $x \in \mathbb{R}^n$ and the output $y \in \mathbb{R}^m$. The input-output relationship is assumed to be linear, i.e., $y = Ax$ with a real $m \times n$-matrix A. Also suppose that the elements of A—parameters of the system—may be varied by our will within some prescribed intervals $\mathbf{a}_{ij}$, so that A can be made any one from the corresponding interval matrix $\mathbf{A} = (\mathbf{a}_{ij})$. In other words, we have a possibility in some way to *control* the parameters of the system in the specified bounds $(\mathbf{a}_{ij})$.

If some rectangular box (interval vector) $\mathbf{y}$ represents the set of output vectors of the considered device, then the question naturally arises of whether there exist input signals x which can be transferred to *any* desired output state $y \in \mathbf{y}$ by the appropriate choice of the parameters (a_{ij}) of the system. The set of all such x (if nonempty) just constitutes $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{y})$. Hence, it makes sense to refer to $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ as the *controllable solution set* of the interval linear algebraic system $\mathbf{A}\, x = \mathbf{b}$, since it distinguishes the set of the inputs x with respect to their controllability properties in the sense of (1).

All the ideas developed above are straightforwardly transferred to the case of general interval nonlinear systems of the form

$$F(x, \mathbf{a}) = \mathbf{b}, \tag{4}$$

where $F(x, a) = (f_1(x, a), f_2(x, a), \dots, f_m(x, a))$ and $\mathbf{x}, \mathbf{a}$ are interval vectors of the same dimension as x, a, respectively. Let us call the set

$$\Sigma_{\exists\forall}(F; \mathbf{a}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall b \in \mathbf{b})(\exists a \in \mathbf{a})(\, F(x, a) = b)\}$$

the *controllable solution set* to the interval system (4). If the input-output relationship for the system has the form $y = F(x, a)$, then the points of

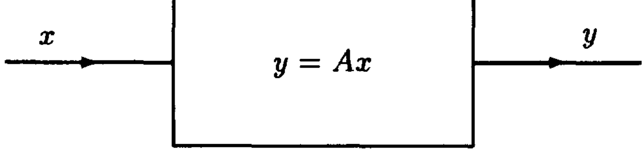

FIG. 1. Model for interpretation of the controllable solution set.

$\Sigma_{\exists\forall}(F; \mathbf{a}, \mathbf{y})$ (and no one else) can be transformed to any required output state $y \in \mathbf{y}$ through some control subjection $a \in \mathbf{a}$.

The controllable solution set seems to appear in the implicit form even in the work by Khlebalin and Shokin [7]. But for the first time the definition (3) had been written out explicitly by Lakeyev and Noskov [8], who gave no name to this set, but examined some of its properties. In particular, they pointed out that

$$\begin{aligned}
&\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b}) \cap \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \\
&\quad = \{\, x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\exists b \in \mathbf{b})(\, Ax = b) \;\& \\
&\qquad\qquad (\forall b \in \mathbf{b})(\exists A \in \mathbf{A})(\, Ax = b)\} \\
&\quad = \{\, x \in \mathbb{R}^n \mid \mathbf{A} \cdot x = \mathbf{b}\}.
\end{aligned} \tag{5}$$

Their main result concerning the set (3) is the following

PROPOSITION 1. [8]

$$\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b}) = \Big\{ x' - x'' \mid x', x'' \in \mathbb{R}^n,\ x', x'' \geqslant 0, \langle x', x'' \rangle = 0, \\ \underline{\mathbf{A}}\, x' - \overline{\mathbf{A}}\, x'' \leqslant \underline{\mathbf{b}}, \overline{\mathbf{A}}\, x' - \underline{\mathbf{A}}\, x'' \geqslant \overline{\mathbf{b}} \Big\},$$

where $\langle \cdot\,, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$ (that is, the sum of products of the components), and $\underline{\mathbf{a}}$, $\overline{\mathbf{a}}$ stand for the lower and upper bounds on an interval (interval vector or matrix) respectively.

To make our considerations more vivid and pictorial, turn to Figure 2, where the controllable solution set to the interval system

$$\begin{pmatrix} [-2, 1] & [-1, 1] \\ [-1, 1] & [-1, 2] \end{pmatrix} x = \begin{pmatrix} [-2, 1] \\ [-1, 2] \end{pmatrix} \tag{6}$$

is depicted. It is the whole of the plane $\mathbb{R}^2$ with the star around the origin of coordinates removed.

The configuration of this picture is typical in some sense, insofar as the zero vector may not belong to $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ unless $\mathbf{b} \subseteq \mathbf{A} \cdot 0 = 0$, i.e., $\mathbf{b} = 0$. Just for this reason the controllable solution set to (6) avoids the origin of

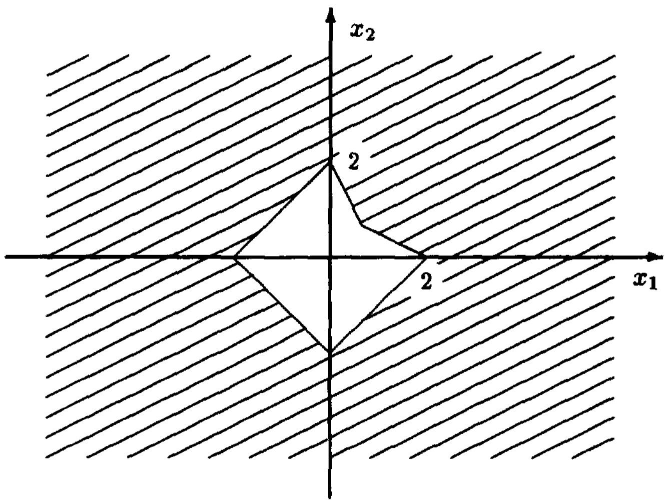

FIG. 2. Controllable solution set to (6).

coordinates in Figure 2. Besides, the above proposition implies that the intersection of $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ with each orthant of $\mathbb{R}^n$ is a polyhedral set, i.e., the intersection of a finite number of half-spaces. In particular, the controllable solution set to interval linear system is always closed.

If the dimensionality of the interval system is large, then the direct description of its controllable solution set becomes laborious and practically useless (its complexity is proportional to $m \cdot 2^n$). For this reason it is sensible to confine ourselves to finding some simple subsets $\Pi \subseteq \Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$, since for any $x \in \Pi$ the determining condition

$$(\forall b \in \mathbf{b})(\exists A \in \mathbf{A})(\ Ax = b)$$

remains valid. Put differently, we change $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ for its *inner approximation.* Since the interval vectors, i.e., the direct products of segments of the real axis, most likely have the simplest structure among the sets in $\mathbb{R}^n$, and so, in conformity with the spirit of the interval analysis, we formulate the following promising interval problem: Find an interval vector that is contained in the controllable solution set (if nonempty) of the interval algebraic system. We shall call this problem the *control problem* for the interval linear algebraic system (2). The powerful computational procedures for its solution have been proposed in [9] recently.

In common with the tolerable solution set, the controllable solution set may turn out to be empty even for "good" interval data, as, for instance, it does in the one-dimensional case $\mathbf{A} = [2,3]$, $\mathbf{b} = [1,2]$. The popular model system

$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} x = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix}$$

from the work of Barth and Nuding [10] gives us the more complex example with the empty controllable solution set.

The main mathematical outcomes of this work are simple sufficient criteria for the controllable solution set of the interval linear algebraic system to be nonempty and for the intersection of this solution set with some orthants to be empty. These are especially important on account of the recent remarkable result from [8]: the problem of recognition whether $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b}) \neq \varnothing$ is NP-complete. But before proceeding, we have to revise some auxiliary notions and facts of the interval analysis.

To begin with, if the i-th row of $\mathbf{A}$ contains only zero elements, then the necessary condition for the controllable solution set to be nonempty is $\mathbf{b}_i = 0$. If so, the property of $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ being empty or nonempty depends upon the other, not the i-th, rows of $\mathbf{A}$ and components of $\mathbf{b}$. Thus, without loss in generality, we may consider in general that $\mathbf{A}$ does not have zero rows.

In our consideration, an important role will play the midpoint (median) and the radius of an interval, i.e., the values

$$\operatorname{mid}\mathbf{p} = (\overline{\mathbf{p}} + \underline{\mathbf{p}})/2,$$

$$\operatorname{rad}\mathbf{p} = (\overline{\mathbf{p}} - \underline{\mathbf{p}})/2.$$

If $\mathbf{p}$ is an interval vector or matrix, these operations should be understood component-wise, so that for $\mathbf{p} = (\mathbf{p}_i)_{i=1}^n$, for example, $\operatorname{mid}\mathbf{p} = (\operatorname{mid}\mathbf{p}_i)_{i=1}^n$. Also denote

$$\sigma(\mathbf{p}) = \operatorname{sign}(\overline{\mathbf{p}} + \underline{\mathbf{p}}).$$

The two intervals $\mathbf{p}$ and $\mathbf{q}$ are called *unidirectional* if $\sigma(\mathbf{p})\sigma(\mathbf{q}) \geqslant 0$.

To characterize "relative narrowness" of nonzero intervals, Ratschek introduced in [11] the functional

$$\chi(\mathbf{p}) = \begin{cases} \underline{\mathbf{p}}/\overline{\mathbf{p}}, & \text{if } |\underline{\mathbf{p}}| \leqslant |\overline{\mathbf{p}}|, \\ \overline{\mathbf{p}}/\underline{\mathbf{p}}, & \text{otherwise.} \end{cases}$$

Clearly, $-1 \leqslant \chi(\mathbf{p}) \leqslant 1$, and $\chi(\mathbf{p}) = 1$ if and only if $\mathbf{p} \in \mathbb{R}$. Moreover, it turns out that

$$\chi(\mathbf{p}) = \chi(\mathbf{q}) \text{ if and only if } \mathbf{p} = t\mathbf{q},\ t \in \mathbb{R},\ t \neq 0, \tag{7}$$

$$\text{if } \mathbf{p} \text{ and } \mathbf{q} \text{ are unidirectional, then } \chi(\mathbf{p} + \mathbf{q}) \geqslant \min\{\chi(\mathbf{p}), \chi(\mathbf{q})\}, \tag{8}$$

$$\text{if } \mathbf{p} + \mathbf{q} \neq 0, \text{ then } \chi(\mathbf{p} + \mathbf{q}) \leqslant \max\{\chi(\mathbf{p}), \chi(\mathbf{q})\}. \tag{9}$$

The proofs of these facts can be found in [11]. Also, sometimes the following obvious property can prove helpful: if $\mathbf{p} \supseteq \mathbf{q}$ and $\chi(\mathbf{q}) \geqslant 0$, then $\chi(\mathbf{p}) \leqslant \chi(\mathbf{q})$. Further we will need a kind of the converse statement:

PROPOSITION 2. *If* $\operatorname{mid}\mathbf{p} = \operatorname{mid}\mathbf{q}$ *and* $-1 < \chi(\mathbf{p}) \leqslant \chi(\mathbf{q})$, *then* $\mathbf{p} \supseteq \mathbf{q}$.

PROOF. Denote $\mu = \operatorname{mid}\mathbf{p} = \operatorname{mid}\mathbf{q}$. If $-1 < \chi(\mathbf{p}) \leqslant \chi(\mathbf{q})$, then $\mu \neq 0$, i.e., $\mu < 0$ or $\mu > 0$. Without loss of generality, we may allow the second opportunity, since the case of negative μ is treated in a similar manner. Under these conditions $|\underline{\mathbf{p}}| < |\overline{\mathbf{p}}|$ and $|\underline{\mathbf{q}}| < |\overline{\mathbf{q}}|$, and so $\chi(\mathbf{p}) \leqslant \chi(\mathbf{q})$ implies

$$\underline{\mathbf{p}}/\overline{\mathbf{p}} \leqslant \underline{\mathbf{q}}/\overline{\mathbf{q}},$$

or

$$\frac{\mu - \operatorname{rad}\mathbf{p}}{\mu + \operatorname{rad}\mathbf{p}} \leqslant \frac{\mu - \operatorname{rad}\mathbf{q}}{\mu + \operatorname{rad}\mathbf{q}}.$$

We obtain from it after simple transformations

$$\mu \cdot \operatorname{rad}\mathbf{p} \geqslant \mu \cdot \operatorname{rad}\mathbf{q},$$

which is equivalent (in view of $\mu > 0$) to $\operatorname{rad}\mathbf{p} \geqslant \operatorname{rad}\mathbf{q}$, i.e., $\mathbf{p} \supseteq \mathbf{q}$. ■

Now we are able to formulate and to prove

THEOREM 1. *Let the interval $m \times n$-matrix* $\mathbf{A}$ *and the interval m-vector* $\mathbf{b}$ *be such that the conditions*

$$\text{(i)} \qquad \mathbf{b}_i \neq 0,$$

$$\text{(ii)} \qquad -1 < \max\left\{ \chi(\mathbf{a}_{ij}) \mid 1 \leqslant j \leqslant n,\, \mathbf{a}_{ij} \neq 0 \right\} \leqslant \chi(\mathbf{b}_i)$$

are valid for any $i \in \{1, 2, \ldots, m\}$. If the "middle system" $\operatorname{mid} \mathbf{A} \cdot x = \operatorname{mid} \mathbf{b}$ *is compatible, then its solutions belong to the controllable solution set* $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ *(which is accordingly nonempty).*

PROOF. If $\tilde{x}$ is a solution to the "middle" point system, then [2]

$$\operatorname{mid}(\mathbf{A}\, \tilde{x}) = (\operatorname{mid} \mathbf{A}) \cdot \tilde{x} = \operatorname{mid} \mathbf{b}.$$

Furthermore, since $-1 < \chi(\mathbf{b}_i)$, $i = 1, 2, \ldots, m$, we have $\operatorname{mid}(\mathbf{A}\, \tilde{x})_i \neq 0$. So, the following calculations are legitimate for each $i \in \{1, 2, \ldots, m\}$

$$\begin{aligned}
\chi((\mathbf{A}\, \tilde{x})_i) &= \chi\left(\sum_{j=1}^{n} \mathbf{a}_{ij} \tilde{x}_j \right) \\
&\leqslant \max\left\{ \chi\left(\mathbf{a}_{ij} \tilde{x}_j\right) \mid 1 \leqslant j \leqslant n,\, \mathbf{a}_{ij} \tilde{x}_j \neq 0 \right\} && \text{by (9)} \\
&= \max\left\{ \chi(\mathbf{a}_{ij}) \mid 1 \leqslant j \leqslant n,\, \mathbf{a}_{ij} \tilde{x}_j \neq 0 \right\} && \text{by (7)} \\
&\leqslant \max\left\{ \chi(\mathbf{a}_{ij}) \mid 1 \leqslant j \leqslant n,\, \mathbf{a}_{ij} \neq 0 \right\}.
\end{aligned}$$

Thus, $-1 < \chi((\mathbf{A}\, \tilde{x})_i) \leqslant \chi(\mathbf{b}_i)$ holds true for $i \in \{1, 2, \ldots, m\}$ and by Proposition 2

$$(\mathbf{A}\, \tilde{x})_i \supseteq \mathbf{b}_i, \qquad i \in \{1, 2, \ldots, m\},$$

which proves Theorem 1. ■

For example, both conditions of Theorem 1 are satisfied for the interval system (6) considered above. The solution to its "middle system" is the vector $(1, 1)^\top$, and it is seen (in Figure 2) to be in the controllable solution set of (6). Alternately, the condition (ii) of Theorem 1 does not hold for the interval linear system

$$\begin{pmatrix} [2,4] & [-2,2] \\ [-2,2] & [2,4] \end{pmatrix} x = \begin{pmatrix} [1,5] \\ [1,5] \end{pmatrix}, \tag{10}$$

while its "midpoint" system is compatible and has the solution $(1,1)^{\top}$ that belongs in the nonempty controllable solution set of (10).

It may appear from the statement of Theorem 1 that the solution of the "middle system" is the most probable representative of the controllable solution set to interval algebraic systems. However, the following counterexample shows that this is not so in general. For the system

$$\begin{pmatrix} 4 & [1,3] \\ [0,2] & 4 \end{pmatrix} x = \begin{pmatrix} [5,7] \\ 4 \end{pmatrix},$$

we have $\Sigma_{\exists\forall} = \{(1,1)^{\top}\}$, but the solution to the "middle system" is $(8/7, 5/7)^{\top}$.

In spite of the apparent unwieldiness of Theorem 1, the implementation of that criterion requires as low as $O(mn)$ of arithmetical and logical operations. The question of compatibility of the "middle system" is simply resolved if $\mathbf{A}$ is nonsingular interval matrix (i.e., contains only nonsingular point matrices). In its turn there are developed numerical algorithms for testing whether an interval matrix is nonsingular [3], although this problem is not trivial on the whole. To summarize, one may assert that the above criterion is quite practical, but not sufficiently sensitive. It is intended for a preliminary rough examination of a given problem.

When solving an actual problem, aside from the solution on its own, one frequently needs some characteristics of its stability that indicate a solvability margin or a measure of the compatible state stability. In our opinion, the magnitude

$$\varrho = \min_{1 \leqslant i \leqslant m} \left\{ \chi(\mathbf{b}_i) - \max\{ \chi(\mathbf{a}_{ij}) \mid 1 \leqslant j \leqslant n,\ \mathbf{a}_{ij} \neq 0 \} \right\}$$

can be applied for this purpose, as a crude quantitative measure of the compatibility factor in the case of $\varrho > 0$.

It is very instructive to correlate the result of Theorem 1 with that of the work [6]. Recall that the following sufficient criterion for the tolerable solution set to be empty is obtained there.

THEOREM 2. *Let the interval* $m \times n$*-matrix* $\mathbf{A}$ *and interval* m*-vector* $\mathbf{b}$ *be such that for some* $k \in \{1, 2, \dots, m\}$ *the conditions hold*

$$\text{(i)} \qquad 0 \notin \mathbf{b}_k,$$

$$\text{(ii)} \qquad \max\left\{ \chi(\mathbf{a}_{kj}) \mid 1 \leqslant j \leqslant n,\ \mathbf{a}_{kj} \neq 0 \right\} < \chi(\mathbf{b}_k).$$

Then the tolerable solution set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ *is empty.*

As is seen, this statement is in the remarkable duality to Theorem 1, while the sets $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ and $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ are in a certain antagonism to each other.

Our next outgrowth is not not so showy as the above theorems, but it may also prove helpful in some cases. Preparatory to formulating it, notice that each orthant in $\mathbb{R}^n$ is completely defined by indicating the signs of components of its inner points, and for this reason we shall speak about the orthants

$$(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n), \tag{11}$$

where $\varepsilon_j = \pm 1$, $j = 1, 2, \dots, n$. If $\varepsilon_l = 0$, then let us agree that the symbol (11) denotes any of the orthants $(\varepsilon_1, \dots, \varepsilon_{l-1}, -1, \varepsilon_{l+1}, \dots, \varepsilon_n)$ or $(\varepsilon_1, \dots, \varepsilon_{l-1}, 1, \varepsilon_{l+1}, \dots, \varepsilon_n)$.

THEOREM 3. *Let the interval* $m \times n$*-matrix* $\mathbf{A}$ *and the interval* m*-vector* $\mathbf{b}$ *be such that*

$$\min\left\{ \chi(\mathbf{a}_{kj}) \mid 1 \leqslant j \leqslant n,\ \mathbf{a}_{kj} \neq 0 \right\} > \chi(\mathbf{b}_k) \geqslant 0$$

for some $k \in \{1, 2, \dots, m\}$. *Then the controllable solution set* $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ *has empty intersection with the orthants* $(\sigma(\mathbf{a}_{k1}), \sigma(\mathbf{a}_{k2}), \dots, \sigma(\mathbf{a}_{kn}))$ *and* $(-\sigma(\mathbf{a}_{k1}), -\sigma(\mathbf{a}_{k2}), \dots, -\sigma(\mathbf{a}_{kn}))$.

PROOF. Proof of this theorem will be conducted *ad absurdum*. Suppose that the condition of Theorem 2 holds for some index k, but there is a point $t \in \Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ from the orthant $(\sigma(\mathbf{a}_{k1}), \sigma(\mathbf{a}_{k2}), \dots, \sigma(\mathbf{a}_{kn}))$. Then $\mathbf{A}\,t \supseteq \mathbf{b}$, and in particular

$$(\mathbf{A}\,t)_k \supseteq \mathbf{b}_k,$$

which is equivalent to

$$\chi\big((\mathbf{A}\,t)_k\big) \leqslant \chi(\mathbf{b}_k), \tag{12}$$

since $\chi(\mathbf{b}_k) \geqslant 0$.

On the other hand, all the intervals $\mathbf{a}_{kj}t_j$, $j = 1, 2, \dots, n$, are unidirectional by the very construction and so

$$\begin{aligned}
\chi\big((\mathbf{A}\,t)_k\big) &= \chi\left(\sum_{j=1}^{n} \mathbf{a}_{kj}t_j\right) \\
&\geqslant \min\big\{\chi\big(\mathbf{a}_{kj}t_j\big) \mid 1 \leqslant j \leqslant n,\ \mathbf{a}_{kj}t_j \neq 0\big\} \qquad \text{by (8)} \\
&= \min\big\{\chi\big(\mathbf{a}_{kj}\big) \mid 1 \leqslant j \leqslant n,\ \mathbf{a}_{kj}t_j \neq 0\big\} \qquad \text{by (7)} \\
&\geqslant \min\big\{\chi\big(\mathbf{a}_{kj}\big) \mid 1 \leqslant j \leqslant n,\ \mathbf{a}_{kj} \neq 0\big\}.
\end{aligned}$$

Hence, it follows from the condition of Theorem 2 that

$$\chi\big((\mathbf{A}\,t)_k\big) > \chi(\mathbf{b}_k)$$

in contradiction to (12). This completes the proof for the orthant $(\sigma(\mathbf{a}_{k1}), \sigma(\mathbf{a}_{k2}), \dots, \sigma(\mathbf{a}_{kn}))$. It is fairly simple to see that it is suitable for the orthant $(-\sigma(\mathbf{a}_{k1}), -\sigma(\mathbf{a}_{k2}), \dots, -\sigma(\mathbf{a}_{kn}))$ too. ■

As a corollary we get

PROPOSITION 3. *Let the interval $m \times n$-matrix* $\mathbf{A}$ *and the interval m-vector* $\mathbf{b}$ *have only nonnegative elements and*

$$\min\big\{\chi(\mathbf{a}_{kj}) \mid 1 \leqslant j \leqslant n, \mathbf{a}_{kj} \neq 0\big\} > \chi(\mathbf{b}_k)$$

for some $k \in \{1, 2, \dots, m\}$. Then the controllable solution set $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ cannot contain nonnegative vectors.

In conclusion we shall develop yet another surprising way of investigating the controllable solution set. As was already noted (5),

$$\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b}) \cap \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid \mathbf{A} \cdot x = \mathbf{b} \,\}, \tag{13}$$

i.e., the intersection of the controllable solution set and the tolerable solution set is the set of all point algebraic solutions to (2). They are very simple to compute. Indeed, taking the median of both sides of the original interval equation (2), we obtain

$$(\mathrm{mid}\,\mathbf{A})\cdot x = \mathrm{mid}\,\mathbf{b}.$$

Hence, to determine the set (13), one only need to solve the middle linear system and to test its solutions for the equality $\mathbf{A}\cdot x = \mathbf{b}$.

Furthermore, the problem of recognition whether $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})$ is empty or not is also not hard. It can be solved efficiently for polynomial time [6]. Thus, if, for instance, $\{x\in\mathbb{R}^n \mid \mathbf{A}\cdot x = \mathbf{b}\} = \varnothing$, whereas $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b}) \neq \varnothing$, then we can conclude that $\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b}) = \varnothing$.

This work was partially supported by Krasnoyarsk Regional Science Foundation, Krasnoyarsk, Russia, under grant 3F0242.

REFERENCES

1 G. Alefeld and J. Herzberger, *Introduction to Interval Computations*, Academic Press, New York, 1983.
2 A. Neumaier, *Interval Methods for Systems of Equations*, Cambridge University Press, Cambridge, 1990.
3 J. Rohn, Systems of linear interval equations. *Linear Algebra Appl.* 126:39–78 (1989).
4 A. Neumaier, Tolerance analysis with interval arithmetic. *Freiburger Intervall-Berichte* 86(9):5–19 (1986).
5 B. Kelling and D. Oelschlägel, Zur Lösung von linearen Toleranzproblemen. *Wiss. Z. TH Leuna-Merseburg* 33(1):121–131 (1991).
6 S. P. Shary, Solving the interval linear tolerance problem. *Math. Comput. Simulation* 39:53–85 (1995).
7 N. A. Khlebalin and Yu. I. Shokin, Interval variant of the modal control method. *Doklady Akademii Nauk* 316(4):846–850 (1991). (in Russian)
8 A. V. Lakeyev and S. I. Noskov, Description of the solution set to linear equation with the intervally defined operator and right-hand side. *Doklady Akademii Nauk* 330(4):430–433 (1993). (in Russian)
9 S. P. Shary, Algebraic approach to the interval linear static identification, tolerance and control problems, or One more application of Kaucher arithmetic. *Reliable Computing* 2:3–33 (1996).
10 W. Barth and E. Nuding, Optimale Lösung von Intervallgleichunssystemen. *Computing*, 12:117–125 (1974).
11 H. Ratschek, Teilbarkeitskriterien der Intervallarithmetik. *J. Reine Angew. Mathematik* 252:128–138 (1972).