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Parallel Interval Multisplittings

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Dedicated to the memory of Peter Henrici

Summary. We introduce interval multisplittings to enclose the set $S = \{A^{-1}b \mid A \in [A], b \in [b]\}$, where $[A]$ denotes an interval matrix and $[b]$ an interval vector. The resulting iterative multisplitting methods have a natural parallelism. We investigate these methods with respect to convergence, speed of convergence and quality of the resulting enclosure for S.

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1 Introduction

Suppose that we are given a linear system

$$Ax = b, \tag{1.1}$$

where A is a nonsingular matrix from $\mathbb{R}^{n\times n}$ and x and b are from $\mathbb{R}^n$. In order to compute the solution x of (1.1) iteratively, O'Leary and White [13] proposed multisplitting methods which are based on several splittings of the matrix A. More precisely, in [13] a *multisplitting* of A is defined as a collection of triples (M_k, N_k, E_k), $k=1, \ldots, K$, such that for all k the matrices M_k, N_k, E_k are from $\mathbb{R}^{n\times n}$, M_k is nonsingular, $A = M_k - N_k$, and E_k is a diagonal matrix with nonnegative entries satisfying $\sum_{k=1}^{K} E_k = I$ ($n \times n$-identity matrix). The corresponding *multisplitting method* to solve (1.1) is given by the iteration

$$x^{m+1} = \sum_{k=1}^{K} E_k y^{m,k}, \quad m=0,1,\ldots, \tag{1.2}$$

where

$$M_k y^{m,k} = N_k x^m + b, \quad k=1,\ldots,K.$$

This multisplitting method has a natural parallelism, since the calculations of $y^{m,k}$ for various k are independent and may therefore be performed in parallel. Moreover, the i-th component of $y^{m,k}$ needs not be computed if the corresponding diagonal entry of E_k is zero. This may result in considerable savings of computational time. We refer to [13] for details. Convergence results for method (1.2) were first given in [13]. Later, Neumann and Plemmons [12] obtained more qualitative results for one of the cases considered in [13].

Now let $I\mathbb{R}^n$ and $I\mathbb{R}^{n\times n}$ denote the set of real interval vectors and of real $n\times n$-interval matrices, respectively. Let $[A]\in I\mathbb{R}^{n\times n}$, $[b]\in I\mathbb{R}^n$ and consider the problem of determining the set

$$S=\{x\in\mathbb{R}^n \mid Ax=b \text{ for some } A\in[A], b\in[b]\}. \tag{1.3}$$

Problems of this kind arise for example if the entries of A and b in (1.1) are not known exactly or in methods for enclosing the solution of a nonlinear system of equations (cf. [2], Chap. 19). Generally, the set S is not an interval vector and it is a very difficult task to calculate S exactly. Therefore, one usually only tries to get an interval vector $[x]$ containing S.

In the present paper we propose and analyze a multisplitting technique to calculate an interval vector $[x]$ containing S. We suppose that the reader is familiar with the elementary rules of real interval arithmetic as described in [2], e.g. In addition, we use the notation $[y]=IGA([B],[c])\in I\mathbb{R}^n$ to denote the interval vector $[y]$ resulting from the interval Gaussian algorithm (cf. [2]) applied to $[B]\in I\mathbb{R}^{n\times n}$ and to the "right hand side" $[c]\in I\mathbb{R}^n$.

Definition 1. Let $[A]\in I\mathbb{R}^{n\times n}$ be nonsingular (i.e. each matrix $A\in\mathbb{R}^{n\times n}$ with $A\in[A]$ is nonsingular). Then the collection of triples $([M]_k,[N]_k,E_k)$, $k=1,\ldots,K$, with $[M]_k,[N]_k\in I\mathbb{R}^{n\times n}$ is called an *interval multisplitting* of $[A]$ if the following three conditions hold:

i) $[A]=[M]_k-[N]_k$ for $k=1,\ldots,K$.

ii) For $k=1,\ldots,K$ the interval Gaussian algorithm is feasible when applied to the matrix $[M]_k$ and any "right hand side" $[c]\in I\mathbb{R}^n$.

iii) For $k=1,\ldots,K$ the matrices E_k are diagonal matrices with nonnegative entries such that $\sum_{k=1}^{K} E_k=I$.

The corresponding *interval multisplitting method* to enclose the set S given by (1.3) is defined by the iteration

$$[x]^{m+1}=\sum_{k=1}^{K} E_k[y]^{m,k}, \qquad m=0,1,\ldots, \tag{1.4}$$

where

$$[y]^{m,k}=IGA([M]_k,[N]_k[x]^m+[b]), \qquad k=1,\ldots,K. \quad \square$$

As method (1.2) the interval multisplitting method (1.4) has a natural parallelism, since the computations of $[y]^{m,k}$ for various k are independent. Again, a compo-

nent of $[y]^{m,k}$ needs not be computed if the corresponding diagonal entry of E_k is zero.

In our paper we will give sufficient conditions on the interval multisplitting $([M]_k, [N]_k, E_k)$ which guarantee the convergence of method (1.4). These conditions also yield a new convergence theorem for the "classical" multisplitting method (1.2). If method (1.4) is convergent it will turn out that its limit $[x]^*$ contains the set S defined by (1.3). We will investigate how close $[x]^*$ is to S and we will derive inequalities which may be interpreted as information on the speed of convergence of interval multisplitting methods.

All these results are contained in Sect. 3, whereas Sect. 2 is devoted to the introduction of notation. In Sect. 4 we report some numerical experiments.

2 Notation

We write interval quantities in square brackets, matrices in capital letters, vectors and scalars in small letters. Without further reference we use the notation $x=(x_i)$, $A=(a_{ij})$ for elements of $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, respectively. By $[\underline{A}, \bar{A}]$, $([a]_{ij})$, $([\underline{a}_{ij}, \bar{a}_{ij}])$ we mean the same interval matrix $[A]$, and we use a similar notation for interval vectors and intervals. Point intervals, i.e. degenerate intervals $[c, c]$, can be identified with the element which they contain; therefore we write c instead of $[c, c]$. Point vectors and point matrices are written in an analogous way. Examples are the zero vector $([0, 0])\equiv 0$ and the identity matrix $[I, I]\equiv I$.

For intervals $[a]$, $[b]$ we define the width, the absolute value and the distance by $d[a]:=\bar{a}-\underline{a}$, $|[a]|:=\max\{|\underline{a}|, |\bar{a}|\}$ and $q([a], [b]):=\max\{|\underline{a}-\underline{b}|, |\bar{a}-\bar{b}|\}$, respectively. For interval vectors and interval matrices these quantities are defined entrywise, e.g., $|[A]|:=(|[a]_{ij}|)$ for $[A]\in I\mathbb{R}^{n\times n}$; $|[A]|$ is a nonnegative real $n\times n$-matrix where nonnegativity refers to the usual entrywise defined partial ordering $\leqq$ with which we equip $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$. We extend this partial ordering to $I\mathbb{R}^{n\times n}$ by setting

$$[A]\leqq[B]:\Leftrightarrow \underline{A}\leqq\underline{B}\wedge\bar{A}\leqq\bar{B}.$$

We write $x>0$ if x is a positive vector, i.e. if all its components are positive, and similarly $A>0$ for positive matrices.

As in [16] a nonsingular matrix $A\in\mathbb{R}^{n\times n}$ is called M-matrix if $a_{ij}\leqq 0$ for $i\neq j$ and if $A^{-1}\geqq 0$. An $n\times n$-interval matrix $[A]$ is termed M-matrix (cf. [3]) if each element $A\in[A]$ is an M-matrix. It is called nonsingular (cf. Definition 1), if this is true for each of its elements A. We sometimes will use the comparison matrix $\langle[A]\rangle=(c_{ij})\in\mathbb{R}^{n\times n}$ of $[A]$ which is defined by

$$c_{ij}:=\begin{cases}\inf\{|a|\,|\,a\in[a]_{ii}\} & \text{if } i=j\\ -|[a]_{ij}| & \text{if } i\neq j.\end{cases}$$

We call $[A]\in I\mathbb{R}^{n\times n}$ an H-matrix if $\langle[A]\rangle$ is an M-matrix [10, 4]. Generalizing a definition of Schneider [14] we use the term M-splitting for the representation $[A]=[M]-[N]\in I\mathbb{R}^{n\times n}$ if $[M]$ is an M-matrix and if $[N]\geqq 0$.

Let $\rho(A)$ denote the spectral radius of a real $n \times n$-matrix A. A function $f: I\mathbb{R}^n \to I\mathbb{R}^n$ is called P-contraction if there exists a nonnegative contraction matrix $P \in \mathbb{R}^{n \times n}$ such that $\rho(P) < 1$ and

$$q(f([x]), f([y])) \leqq P q([x], [y])$$

for all $[x], [y] \in I\mathbb{R}^n$. For a P-contraction f it is known [2, 8] that it has a unique fixed point $[x]^* \in I\mathbb{R}^n$ and that the sequences $\{[x]^m\}$ defined by the iterative method

$$[x]^{m+1} = f([x]^m), \quad m = 0, 1, \ldots, \tag{2.1}$$

are convergent to $[x]^*$ for any starting vector $[x]^0 \in I\mathbb{R}^n$. Furthermore $\rho(P)$ is an upper bound for the R_1-factor $R_1((2.1), [x]^*)$ of (2.1) which is defined as in [2] by

$$R_1((2.1), [x]^*) := \sup\{ \lim_{m \to \infty} \sup \|q([x]^m, [x]^*)\|^{1/m} \mid \{[x]^m\} \in \mathscr{C}\} \tag{2.2}$$

($\|\cdot\|$ any vector norm of $\mathbb{R}^n$; $\mathscr{C}$ set of all sequences constructed by (2.1) and converging to $[x]^*$).

3 Results

We start this section by a fundamental theorem presenting a class of interval matrices $[A]$ and splittings $[M]_k - [N]_k$ for which the feasibility of method (1.4) can be guaranteed.

Theorem 1. *Let* $([M]_k, [N]_k, E_k)$, $k = 1, \ldots, K$, *be an interval multisplitting of* $[A] \in I\mathbb{R}^{n \times n}$ *with* $[M]_k$ *being H-matrices. Define* $\tilde{A}_k := \langle [M]_k \rangle - |[N]_k|$, $k = 1, \ldots, K$, *and let the solution set* S *be defined as in* (1.3). *If there is a positive vector* $x \in \mathbb{R}^n$ *such that* $\tilde{A}_k x > 0$ *for all* k, *then the following assertions hold:*

a) *For any starting vector* $[x]^0 \in I\mathbb{R}^n$ *method* (1.4) *is feasible. Each sequence of iterates constructed by* (1.4) *converges to a common limit* $[x]^*$.

b) *Method* (1.4) *is inclusion isotone, i.e., if* $[x]^0 \subseteq [y]^0$ *then* $[x]^m \subseteq [y]^m$, $m = 0, 1, \ldots$.

c) $S \subseteq [x]^0$ *implies* $S \subseteq [x]^m$, $m = 0, 1, \ldots$.

d) $S \subseteq [x]^*$.

e) $R_1((1.4), [x]^*) \leqq \rho(P) < 1$, *where* $P := \sum_{k=1}^{K} E_k \langle [M]_k \rangle^{-1} |[N]_k|$. □ (3.1)

To prove Theorem 1 we need the following auxiliary result.

Lemma 1. *Let* $[b]\in I\mathbb{R}^n$, $[M]$, $[N]\in I\mathbb{R}^{n\times n}$, $[M]$ *H-matrix. Let the function* f: $I\mathbb{R}^n\to I\mathbb{R}^n$ *be defined by*

$$f([x]):=IGA([M],[N][x]+[b]).$$

Then

$$q(f([x]),f([y]))\leqq\langle[M]\rangle^{-1}|[N]|\,q([x],[y]) \quad \text{for any } [x],[y]\in I\mathbb{R}^n. \tag{3.2}$$

Proof of Lemma 1. Since $[M]$ is an H-matrix, $f([x])$ is defined for any $[x]\in I\mathbb{R}^n$ (cf. [2]). Formula (3.2) can now easily be proved using Schwandt's representation of the interval Gaussian algorithm [15, 1] and by applying Lemma 2c of [6] with the index set $P=\emptyset$ and with $[N][x]+[b]$ instead of $[b]$. □

Proof of Theorem 1

a), e) Since $[M]_k$ are H-matrices the feasibility of (1.4) is guaranteed for any starting vector $[x]^0\in I\mathbb{R}^n$ (cf. [2]).

We show that

$$f([x]):=\sum_{k=1}^{K}E_k\,g_k([x]) \tag{3.3}$$

with

$$g_k([x]):=IGA([M]_k,[N]_k[x]+[b]),\qquad k=1,\ldots,K, \tag{3.4}$$

is a P-contraction.

By the usual rules for the distance q (see [2], e.g.) and by Lemma 1 one gets

$$q(f([x]),f([y]))\leqq\sum_{k=1}^{K}E_k\,q(g_k([x]),g_k([y]))\leqq P\,q([x],[y])$$

with $P\geqq 0$ defined in (3.1). Now

$$Px=\left(I-\sum_{k=1}^{K}E_k\langle[M]_k\rangle^{-1}\tilde{A}_k\right)x<x,$$

since $\tilde{A}_k x>0$ and $\sum_{k=1}^{K}E_k=I$. Therefore

$$\frac{(Px)_i}{x_i}<1,\qquad i=1,\ldots,n,$$

and Exercise 2 in [16, p. 47] guarantees $\rho(P)<1$. Hence f is a P-contraction with contraction matrix P. This proves a) and e).

b) Follows as usually from the inclusion monotonicity of interval arithmetic.

c), d) Let $x^* \in S \subseteq [x]^m$. There exist $A \in [A]$, $b \in [b]$ such that $Ax^* = b$. Choose $M_k \in [M]_k$, $N_k \in [N]_k$ satisfying $M_k - N_k = A$, $k = 1, \ldots, K$. Then $x^* = M_k^{-1}(N_k x^* + b)$, $k = 1, \ldots, K$, implies

$$x^* = \sum_{k=1}^{K} E_k M_k^{-1}(N_k x^* + b) \in f(x^*) \subseteq f([x]^m) = [x]^{m+1}.$$

This proves c), and d) follows from c) by starting with $[x]^0 \supseteq S$ and by taking the limit $m \to \infty$. □

If $[M]_k - [N]_k$ is an M-splitting of $[A]$ we have $\langle [M]_k \rangle = \underline{M}_k$, $|[N]_k| = \overline{N}_k$, $\tilde{A}_k = \underline{A}$. Suppose $\underline{A}^{-1} \geqq 0$ and let $e := (1, 1, \ldots, 1)^T \in \mathbb{R}^n$. Then $x := \underline{A}^{-1} e > 0$ and $\underline{A} x = e > 0$.

Similarly, if $\tilde{A} := \inf\{\tilde{A}_k \mid k = 1, \ldots, K\}$ (the infimum applied entrywise) has a nonnegative inverse, then $\tilde{A}_k(\tilde{A}^{-1} e) > 0$, $k = 1, \ldots, K$.

Thus Theorem 1 implies the following two corollaries.

Corollary 1. *Let $([M]_k, [N]_k, E_k)$, $k = 1, \ldots, K$, be an interval multisplitting of $[A] \in I\mathbb{R}^{n \times n}$ with $[M]_k$ being H-matrices. Define $\tilde{A}_k$ as in Theorem 1 and let $\tilde{A} := \inf\{\tilde{A}_k \mid k = 1, \ldots, K\}$. If $\tilde{A}^{-1} \geqq 0$ then the assertions of Theorem 1 hold.* □

Corollary 2. *Let $([M]_k, [N]_k, E_k)$, $k = 1, \ldots, K$, be an interval multisplitting of $[A] \in I\mathbb{R}^{n \times n}$. Let $[M]_k - [N]_k$, $k = 1, \ldots, K$, be M-splittings of $[A]$ and let $\underline{A}^{-1} \geqq 0$ (e.g., $[A]$ M-matrix). Then the assertions of Theorem 1 hold with* $P = \sum_{k=1}^{K} E_k \underline{M}_k^{-1} \overline{N}_k$. □

We remark that in the point case – i.e., if $\underline{A} = \overline{A}$ and $\underline{b} = \overline{b}$ – the matrices $[M]_k$ and $[N]_k$ must necessarily be point matrices since

$$0 = d[A] = d[M]_k + d[N]_k.$$

Starting the iteration (1.4) by a point vector x^0 results in the classical multisplitting method (1.2) because $IGA(M, Nx + b) = M^{-1}(Nx + b)$, if $M, N \in \mathbb{R}^{n \times n}$, $x, b \in \mathbb{R}^n$. Thus Theorem 1 and Corollary 1 contain new criteria for the convergence of (1.2). In this case the matrices in Corollary 2 reduce to a special class of matrices treated in [13, Theorem 1 a].

In our next two theorems we take a closer look to the quality of the enclosure $[x]^*$ of the solution set S. In particular, we show that for a situation similar to that of Corollary 2 the limit $[x]^*$ of (1.4) is at least as good as the worst enclosure obtained by the iterative methods

$$[x]^{m+1} = IGA([M]_k, [N]_k [x]^m + [b]), \quad m = 0, 1, \ldots \ (k \in \{1, \ldots, K\}). \tag{3.5}$$

Theorem 2. *Let $([M]_k, [N]_k, E_k)$, $k = 1, \ldots, K$, be an interval multisplitting of $[A] \in I\mathbb{R}^{n \times n}$. Let $[M], [N] \in I\mathbb{R}^{n \times n}$ and let the following three conditions hold:*

(i) *$[A]$ is an M-matrix.*

(ii) *$[M] - [N]$ and $[M]_k - [N]_k$, $k = 1, \ldots, K$, are M-splittings of $[A]$.*

(iii) $[M] \leqq [M]_k$ *and* $d[M] \geqq d[M]_k$, $k=1, \ldots, K$ *or – equivalently –* $[N] \leqq [N]_k$ *and* $d[N] \leqq d[N]_k$, $k=1, \ldots, K$.

Then the limit $[x]^*$ *of* (1.4) *is contained in the limit* $[\hat{x}]^*$ *of the iterative method*

$$[\hat{x}]^{m+1} = IGA([M], [N][\hat{x}]^m + [b]), \quad m=0, 1, \ldots . \tag{3.6}$$

Proof. The equivalence in (ii) follows at once by $\underline{M} - \bar{N} = \underline{A} = \underline{M}_k - \bar{N}_k$, $\bar{M} - \underline{N} = \bar{A} = \bar{M}_k - \underline{N}_k$ and $d[M] + d[N] = d[A] = d[M]_k + d[N]_k$.

Let f, g_k be defined as in (3.3), (3.4). Let $[x]_k^*$ be the limit of (3.5) which exists by the assumptions of the theorem. (Use Corollary 2, e.g., with $K=1$.) Satz 10 in [7] guarantees $[x]_k^* \subseteq [\hat{x}]^*$, its proof shows $[\hat{x}]^* \supseteq g_k([\hat{x}]^*)$, $k=1, \ldots, K$. Therefore

$$f([\hat{x}]^*) = \sum_{k=1}^{K} E_k g_k([\hat{x}]^*) \subseteq \sum_{k=1}^{K} E_k [\hat{x}]^* = [\hat{x}]^*.$$

Hence starting (1.4) with $[x]^0 = [\hat{x}]^*$ yields to $[x]^1 \subseteq [\hat{x}]^*$, and iteratively we get $[x]^m \subseteq [\hat{x}]^*$, $m=0, 1, \ldots$. Taking the limit $m \to \infty$ finally results in $[x]^* \subseteq [\hat{x}]^*$. □

Choosing $[M] = [A]$, $[N] = 0$ method (3.6) reduces to the ordinary interval Gaussian algorithm, the iteration in (3.6) being superfluous, of course. In this case condition (iii) of Theorem 2 holds automatically when (i) and (ii) are fulfilled. Thus for M-matrices $[A]$ and M-splittings $[M]_k - [N]_k$ the interval vector $IGA([A], [b])$ always contains the limit $[x]^*$ of (1.4).

The following corollary is an immediate consequence of Theorem 2.

Corollary 3. *Let the assumptions of Theorem 2 hold with* $[M] = [M]_{k_0}$, $[N] = [N]_{k_0}$ *for some* $k_0 \in \{1, \ldots, K\}$. *Then the limit* $[x]^*$ *of* (1.4) *is contained in the limit* $[x]_{k_0}^*$ *of* (3.5). □

While Theorem 2 does not relate $[x]^*$ to the solution set S directly, our next theorem lists conditions which guarantee the best enclosure being possible.

Theorem 3. *Let* $([M]_k, [N]_k, E_k)$, $k=1, \ldots, K$, *be an interval multisplitting of* $[A] \in I\mathbb{R}^{n \times n}$ *and let the following three conditions hold:*

(i) $[A]$ *is an M-matrix.*

(ii) $[M]_k - [N]_k$, $k=1, \ldots, K$, *are M-splittings of* $[A]$.

(iii) $[M]_k$ *is a lower triangular matrix or a point matrix for* $k=1, \ldots, K$.

Then the limit $[x]^*$ *of* (1.4) *is the interval hull* $[x]^H$ *of the solution set* S, *i.e.* $[x]^* = [x]^H := [\inf S, \sup S]$.

Proof. The assumptions of the theorem guarantee that the limits $[x]_k^*$ of (3.5) are equal to $[x]^H$ [7, 11]. Using the notation of (3.3) and (3.4) one therefore gets

$$f([x]^H) = \sum_{k=1}^{K} E_k g_k([x]^H) = \sum_{k=1}^{K} E_k [x]^H = [x]^H.$$

Since the fixed point $[x]^*$ of f is unique (Theorem 1) the theorem is proved. □

Unfortunately, Theorem 3 can become false if $[M]_k$ does not fulfill condition (iii). This can be seen using $K=1$; see Sect. 4 or [8, §4]. Nevertheless one can show that $[x]^*$ yields the hull of S if $[b]$ has a special form.

Theorem 4. *Let the assumptions of Theorem 3 hold replacing* (iii) *by*

(iii') $[b] \geqq 0$ *or* $[b] \leqq 0$ *or* $0 \in [b]$.

Then the limit $[x]^*$ *of* (1.4) *is the interval hull of the solution set S.*

Proof. Let f, g_k be defined as in (3.3), (3.4).

Assuming $[b] \geqq 0$ and setting $[y] := [\bar{A}^{-1}\underline{b}, \underline{A}^{-1}\bar{b}] \geqq 0$ one gets $[N]_k[y] + [b] \geqq 0$. This yields to

$$g_k([y]) = [\bar{M}_k^{-1}(\underline{N}_k \underline{y} + \underline{b}), \underline{M}_k^{-1}(\bar{N}_k \bar{y} + \bar{b})]$$

by a result of Barth and Nuding [3]. Hence

$$g_k([y]) = [\bar{M}_k^{-1}\{(\bar{M}_k - \bar{A})\bar{A}^{-1}\underline{b} + \underline{b}\}, \underline{M}_k^{-1}\{\underline{M}_k - \underline{A})\underline{A}^{-1}\bar{b} + \bar{b}\}] = [y],$$

and Theorem 4 follows for $0 \leqq [b]$ with $[x]^* := [y] = [x]^H$.

The remaining two cases can be proved analogously using $[x]^* := [\underline{A}^{-1}\underline{b}, \bar{A}^{-1}\bar{b}]$ and $[x]^* := \underline{A}^{-1}[b]$, respectively. □

Our final results concern the rate of convergence of method (1.4) compared with that of standard methods (3.6) based on M-splittings. (Among these standard methods are the Jacobi- and the Gauß-Seidel iterative process, e.g.) As in many other iterative methods in interval analysis the R_1-factor of (1.4) is rarely known explicitly and one has to undertake enormous efforts to calculate it exactly (see [9], e.g.). Therefore one often contents oneself with an upper bound. For method (1.4) applied to M-matrices $[A]$ and M-splittings $[M]_k - [N]_k$ we will use the contraction matrix

$$P = \sum_{k=1}^{K} E_k \underline{M}_k^{-1} \bar{N}_k$$

of Corollary 2 as a measure for the rate of convergence. We will give conditions which guarantee that $\rho(P)$ is not greater than the bound $\rho(\underline{M}^{-1}\bar{N})$ for the R_1-factor of the standard methods (3.6).

Theorem 5. *Let* $([M]_k, [N]_k, E_k)$, $k=1, \ldots, K$, *be an interval multisplitting of* $[A] \in I\mathbb{R}^{n\times n}$. *Let* $[M], [N] \in I\mathbb{R}^{n\times n}$ *and let the following three conditions hold.*

(i) $[A]$ *is an M-matrix.*

(ii) $[M]-[N]$ *and* $[M]_k - [N]_k$, $k=1, \ldots, K$, *are M-splittings of* $[A]$.

(iii) $\underline{M}_k \leqq \underline{M}$ *or – equivalently –* $\bar{N}_k \leqq \bar{N}$, $k=1, \ldots, K$.

Then $\rho(P) \leqq \rho(\underline{M}^{-1}\bar{N})$ *where* $P := \sum_{k=1}^{K} E_k \underline{M}_k^{-1} \bar{N}_k$.

Proof. The equivalence in (iii) follows by an argument analogous to that in Theorem 2. Let $e := (1, 1, \ldots, 1)^T \in \mathbb{R}^n$, $\varepsilon > 0$, $N_\varepsilon := \bar{N} + \varepsilon e e^T$. Since $\underline{M} - \bar{N}$ is a regular splitting of the M-matrix $\underline{A}$ we have $\rho(\underline{M}^{-1}\bar{N}) < 1$ (cf. [16]), and by the continuity of ρ the inequality $\rho_\varepsilon := \rho(\underline{M}^{-1} N_\varepsilon) < 1$ holds for all sufficiently small $\varepsilon > 0$. By construction $\underline{M}^{-1} N_\varepsilon$ is positive, therefore the Theorem of Perron and Frobenius guarantees a positive eigenvector x_ε associated with the eigenvalue $\rho_\varepsilon > 0$. Now $\underline{M}^{-1} N_\varepsilon x_\varepsilon = \rho_\varepsilon x_\varepsilon$ implies

$$\rho_\varepsilon \underline{M} x_\varepsilon = N_\varepsilon x_\varepsilon > 0 \quad \text{and} \quad \underline{A} x_\varepsilon \geqq (\underline{M} - N_\varepsilon) x_\varepsilon = (1 - \rho_\varepsilon) \underline{M} x_\varepsilon > 0. \tag{3.7}$$

Taking $\underline{M}_k \leqq \underline{M}$ into account one gets $\underline{M}^{-1} \leqq \underline{M}_k^{-1}$ and

$$0 \leqq P x_\varepsilon = \sum_{k=1}^{K} E_k \underline{M}_k^{-1} \bar{N}_k x_\varepsilon = \sum_{k=1}^{K} E_k \underline{M}_k^{-1} (\underline{M}_k - \underline{A}) x_\varepsilon$$

$$= x_\varepsilon - \sum_{k=1}^{K} E_k \underline{M}_k^{-1} \underline{A} x_\varepsilon \leqq x_\varepsilon - \sum_{k=1}^{K} E_k \underline{M}^{-1} \underline{A} x_\varepsilon \underset{(3.7)}{\leqq} \rho_\varepsilon x_\varepsilon.$$

Hence $0 \leqq \dfrac{(P x_\varepsilon)_i}{(x_\varepsilon)_i} \leqq \rho_\varepsilon$, $i = 1, \ldots, n$, which implies $\rho(P) \leqq \rho_\varepsilon$ by Exercise 2 in [16, p. 47]. Taking the limit $\varepsilon \to 0$ proves the assertion. □

We remark that in the point case Theorem 5 generalizes Theorem 3.1 in [12].

Our final corollary sharpens the result of Theorem 5 by replacing the upper bounds on the R_1-factors by these factors themselves. To formulate it we define the index set $S_{[A]}$ by

$$j \in S_{[A]} :\Leftrightarrow \text{there exists an index } i \in \{1, \ldots, n\} \text{ such that } d[a]_{ij} > 0.$$

Obviously the elements of $S_{[A]}$ mark those columns of $[A]$ which contain at least one nondegenerate interval entry.

Corollary 4. *Let – in addition to the assumptions of Theorem 5 – the following conditions be fulfilled:*

(i) $\underline{A}$ *is irreducible.*

(ii) $[M]$ *is a lower triangular matrix.*

(iii) $(\underline{A}^{-1} \underline{b})_s \leqq 0$ *for all indices* $s \in S_{[A]}$ *or* $(\underline{A}^{-1} \bar{b})_t \geqq 0$ *for all indices* $t \in S_{[A]}$.

Then $R_1((1.4), [x]^*) \leqq R_1((3.6), [\hat{x}]^*)$.

Proof. By a result in [9] the assumptions guarantee $\rho(\underline{M}^{-1}\bar{N}) = R_1((3.6), [\hat{x}]^*)$ and the assertion follows by Theorem 5 and by Theorem 1e). □

We end this section by a remark concerning the degenerate multisplitting case $K = 1$. In this case method (1.4) reduces to the standard method

$$[x]^{m+1} = IGA([M]_1, [N]_1 [x]^m + [b]), \qquad m = 0, 1, \ldots.$$

Thus most of our results are generalizations of theorems in [7–9].

4 Example

Let $[A]=([a]_{ij})$ be a symmetric 24×24-interval matrix having five non-zero diagonals. Let the non-zero off-diagonal entries of [A] be given by

$$[a]_{ij}=\begin{cases}[-1,0] & \text{if } j=i+8 \text{ or } i=j+8\\ \left[-2-\dfrac{i}{6},-1\right] & \text{if } j=i+1\\ \left[-2-\dfrac{j}{6},-1\right] & \text{if } i=j+1\end{cases}$$

and the diagonal entries $[a]_{ii}=[\underline{a}_{ii},\bar{a}_{ii}]$ by

$$\underline{a}_{ii}=2+\sum_{\substack{j=1\\ j\neq i}}^{24}(-\underline{a}_{ij}),\qquad \bar{a}_{ii}=\underline{a}_{ii}+\frac{i}{6}.$$

Let the "right hand side" $[b]\in I\mathbb{R}^{24}$ be given by

$$[b]_i=\begin{cases}[-3,-1] & \text{if } 1\leqq i\leqq 11\\ [1,3] & \text{if } 12\leqq i\leqq 24.\end{cases}$$

In the sequel we will describe several interval multisplittings $([M]_k,[N]_k,E_k)$, $k=1,\ldots,K$, for $[A]$, by writing down the matrices $[M]_k$ and E_k. The matrices $[N]_k$ are then defined in a unique manner by the equation $[A]=[M]_k-[N]_k$. All of the considered splittings $[A]=[M]_k-[N]_k$ are M-splittings of the M-matrix $[A]$. This can easily be seen from the fact that if $B\in[A]$ or $B\in[M]_k$, the matrix B has the sign pattern of an M-matrix and is strictly diagonally dominant with positive diagonal entries (see Chap. 6 of [4]).

a) Let $K=5$. For $k=1,\ldots,5$ the matrix $[M]_k$ is given entrywise by

$$([M]_k)_{ij}=\begin{cases}[a]_{ij} & \text{if } i=j \text{ or } 4(k-1)+1\leqq i,j\leqq 4(k-1)+8\\ 0 & \text{else.}\end{cases}$$

For $k=2,3,4$ the matrix E_k is given by

$$(E_k)_{ii}=\begin{cases}\frac{1}{2} & \text{if } 4(k-1)+1\leqq i\leqq 4(k-1)+8\\ 0 & \text{else,}\end{cases}$$

whereas the matrices E_1 and E_5 are given by

$$(E_1)_{ii}=\begin{cases}1 & \text{if } 1\leqq i\leqq 4\\ \frac{1}{2} & \text{if } 5\leqq i\leqq 8\\ 0 & \text{else,}\end{cases}$$

$$(E_5)_{ii}=\begin{cases}1 & \text{if } 21\leqq i\leqq 24\\ \frac{1}{2} & \text{if } 17\leqq i\leqq 20\\ 0 & \text{else.}\end{cases}$$

This multisplitting is based on an overlapping block scheme for the matrix $[A]$ consisting of the five diagonal blocks given by

$$\{[a]_{ij} | 4(k-1)+1 \leqq i,j \leqq 4(k-1)+8\}, \quad k=1,\ldots,5.$$

It is important to notice that for this interval multisplitting the computation of $[x]^{m+1}$ according to (1.4) requires rather few work: When calculating $[y]^{m,k} = IGA([M]_k, [N]_k[x]^m + [b])$ we actually need only the 8 components of $[y]^{m,k}$ for which $(E_k)_{ii}$ is non-zero. This in turn is equivalent to applying the interval Gaussian algorithm to a smaller system of size 8×8 consisting of the corresponding diagonal block of $[M]_k$. Moreover, this diagonal block is tridiagonal. Thus, as in the point case, considerable additional savings are possible if the interval Gaussian algorithm is implemented appropriately.

b) Let $K=5$. For $k=1,\ldots,5$ take

$$([M]_k)_{ij} = \begin{cases} \bar{a}_{ij} & \text{if } i=j \text{ or } 4(k-1)+1 \leqq i,j \leqq 4(k-1)+8 \\ 0 & \text{else,} \end{cases}$$

and define the matrices E_k, $k=1,\ldots,K$, as in a).

This multisplitting allows the same savings in computational work as in a).

c) Let $K=5$. For $k=1,\ldots,5$ choose

$$([M]_k)_{ij} = \begin{cases} [a]_{ij} & \text{if } i=j \text{ or } 4(k-1)+1 \leqq j \leqq i \leqq 4(k-1)+8 \\ 0 & \text{else.} \end{cases}$$

The matrices E_k, $k=1,\ldots,5$, are given as in a).

For this multisplitting the matrices $[M]_k$ are bidiagonal and lower triangular. This means that the result of the interval Gaussian algorithm may now be obtained very simply by successive straightforward computation of the components of $[y]^{m,k}$ (see Satz 3 in [7]).

d) Let $K=1$ and

$$([M]_1)_{ij} = \begin{cases} [a]_{ij} & \text{if } 1 \leqq j \leqq i \leqq 24 \\ 0 & \text{else,} \end{cases}$$

$E_1 = I$. This is the usual interval version of the Gauß-Seidel method (see [2]).

e) Let $K=1$ and

$$([M]_1)_{ij} = \begin{cases} [a]_{ij} & \text{if } 1 \leqq i=j \leqq 24 \\ 0 & \text{else,} \end{cases}$$

$E_1 = I$. This is the usual interval version of the Jacobi method (see [2]).

f) Let $K=1$ and $[M]=[A]$, $E_1=I$. This (trivial) multisplitting is no longer an iterative method. It merely produces the result $IGA([A]], [b])$.

Our numerical experiments were performed on a serial personal computer kws EB 68/20 using the MC 68020 micro-processor. The programming language

Table 1

	a	b	c	d	e	f
m	31	115	67	57	106	-
$\underline{x}_{24}^m$	−0.4965	−0.3986	−0.3986	−0.3986	−0.3986	−0.5177

was PASCAL-SC (see [5]). PASCAL-SC is an extension of PASCAL which allows in particular an easy handling of interval operations including directed roundings. For all multisplittings a)–f) the starting vector was the zero vector. We stopped the iteration when the inequalities

$$|\underline{x}_i^m - \underline{x}_i^{m-1}| \leqq |\underline{x}_i^{m-1}| \cdot 10^{-10}$$

and

$$|\bar{x}_i^m - \bar{x}_i^{m-1}| \leqq |\bar{x}_i^{m-1}| \cdot 10^{-10}$$

held simultaneously for all $i = 1, \ldots, 24$. The first row of Table 1 reports the number m of iterations necessary to fulfill our stopping criterion. These numbers yield information on the quality of the corresponding interval multisplitting. In particular, comparing these numbers for a) and c) with e) may be regarded as an illustration of Theorem 5.

The second row of Table 1 reports the value of $\underline{x}_{24}^m$ rounded to four digits, where m is the number of necessary iterations. The values for $\underline{x}_{24}^m$ vary with the interval multisplittings used, illustrating Theorems 2 and 3. The value of $\bar{x}_{24}^m$ was 0.7736 for all multisplittings.

The multisplittings b)–e) converge to the same fixed point which is the interval hull of the solution set S defined in (1.3) (see Theorem 3). The multisplitting c) merits particular attention: It requires only slightly more iterations than the Gauß-Seidel method d), but in contrast to d) it has a natural parallelism. The computation of $[x]^{m+1}$ according to (1.4) breaks up into 5 parallel subtasks for computing $[y]^{m,k}$ and as was pointed out earlier, these subtasks are lower triangular bidiagonal systems of size 8×8. One step of the Gauß-Seidel method d), however, is equivalent to solving a lower triangular system of size 24 having 3 non-zero diagonals.

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