Computing 46, 265–274 (1991)

Computing

Interval Linear Systems with Symmetric Matrices, Skew-Symmetric Matrices and Dependencies in the Right Hand Side

C. Jansson, Hamburg-Harburg

Received March 5, 1990

Abstract — Zusammenfassung

Interval Linear Systems with Symmetric Matrices, Skew-Symmetric Matrices and Dependencies in the Right Hand Side. The methods of Interval Arithmetic permit to calculate guaranteed a posteriori bounds for the solution set of problems with interval input data. At present, these methods assume that all input data vary independently between their given lower and upper bounds. This paper shows for special interval linear systems how to handle the case where dependencies of the input data occur.

AMS Subject Classifications: 65G10, 65H10

Key words: Interval arithmetic, systems of equations, data dependencies

Lineare Intervallgleichungssysteme mit symmetrischen Matrizen, schief-symmetrischen Matrizen und Abhängigkeiten in der rechten Seite. Die Intervallarithmetik erlaubt für verschiedene Problemstellungen die Berechnung von a posteriori Schranken für die zugehörige Lösungsmenge; dabei ist stets vorausgesetzt, daß alle Eingabedaten unabhängig voneinander zwischen vorgegebenen unteren und oberen Schranken variieren. Diese Veröffentlichung behandelt Methoden für spezielle lineare Intervallsysteme, die Abhängigkeiten der Eingabedaten mit berücksichtigen.

1. Introduction

In many practical applications a linear algebraic system has to be solved the coefficients of which are uncertain due to measurements. Moreover, frequently the system matrix has a special structure and the coefficients of the matrix or the right hand side may be dependent in some sense. For example if the system matrix A is symmetric the coefficients $a_{\mu\nu}$ are dependent with $a_{\mu\nu} = a_{\nu\mu}$.

Interval mathematics is an elegant tool for working with uncertain data, replacing the input data by real compact *intervals*

$$[a] := [\underline{a}, \overline{a}] := \{a \in \mathbb{R} \mid \underline{a} \leq a \leq \overline{a}\} \tag{1.1}$$

where $\underline{a} \leq \overline{a}$. The values $\underline{a}$ resp. $\overline{a}$ are called *lower* resp. *upper bounds.* $\mathbb{IR}$ denotes the set of all real compact intervals. We assume that the reader is familiar with basic results of this theory (cf. [1], [2], [12], [14], [15]).

We will use the following notations. $m([a]) := \underline{a} + 0.5 \cdot (\overline{a} - \underline{a})$ is the *midpoint* and $d([a]) := \overline{a} - \underline{a}$ is the *diameter* of $[a]$. By $\mathbb{R}^n$ resp. $\mathbb{R}^{n \times n}$ we denote the space of real vectors x with n components resp. the space of real $n \times n$ matrices A. $\mathbb{IR}^n$ is the set of *interval vectors* $[x]$

$$[x] := [\underline{x}, \overline{x}] := \{x \in \mathbb{R}^n | \underline{x} \le x \le \overline{x}\}, \qquad \underline{x} \le \overline{x} \tag{1.2}$$

with n components and $\mathbb{IR}^{n \times n}$ is the set of *interval matrices*

$$[A] := [\underline{A}, \overline{A}] := \{A \in \mathbb{R}^{n \times n} | \underline{A} \le A \le \overline{A}\}, \qquad \underline{A} \le \overline{A} \tag{1.3}$$

with n^2 real interval coefficients $[a_{ij}]$. Diameter and midpoint of interval vectors and interval matrices are defined componentwise. For $X \subseteq \mathbb{R}^n$ the *interval hull* $\diamondsuit(X)$ is defined by

$$\diamondsuit(X) := \bigcap \{[w] \in \mathbb{IR}^n | X \subseteq [w]\}. \tag{1.4}$$

We use $* \in \{+, -, \cdot, /\}$ for the real arithmetic operations and interval operations; from the context it will always be clear whether it is an operation on reals or on intervals.

A *linear interval system* is defined as a family of linear systems

$$Ax = b \tag{1.5}$$

with $A \in [A] \in \mathbb{IR}^{n \times n}$ and $b \in [b] \in IR^n$. Each coefficient $a_{ij} \in [a_{ij}]$, $b_i \in [b_i]$ for $i, j = 1, \ldots, n$ is allowed to vary between the given lower and upper bound independently of the variation of the other coefficients. To "solve" this family of linear systems means to calculate (on a computer) an interval vector $[x]$ containing the *solution set*

$$\Sigma([A], [b]) := \{x \in \mathbb{R}^n | Ax = b, A \in [A], b \in [b]\}. \tag{1.6}$$

There are several methods for computing such interval vectors (cf. [4], [5], [11], [16], [17], [18], [21], [22]). The algorithms of Rump are implemented in a number of commercial and non-commercial packages like ABACUS, ACRITH, CALCULUS, FORTRAN-SC, PASCAL-SC, TPX (cf. [6], [8], [9], [10], [19], [3], [13], [7]).

If dependencies of the input data of $[A]$ or $[b]$ occur the corresponding solution set has to be defined according to those dependencies. In the example case of a symmetric matrix with uncertain data the solution set

$$\{x \in \mathbb{R}^n | Ax = b, A \in [A], b \in [b], A \text{ symmetric}\}$$

is of interest. Obviously this set is contained in $\Sigma([A], [b])$ but may be very small compared to the latter. At present, no special algorithms are known for calculating sharp bounds for such solution sets with dependent interval input data (compare [15], chapter 3.4).

In this paper we present algorithms computing very sharp bounds for the solution sets of linear systems with symmetric resp. skew-symmetric matrices with interval input data and dependencies in the right hand side. They are based on the inclusion methods of Rump ([17], [18], [20]) and permit to estimate the sharpness of the calculated bounds. It is shown how with the tools of Interval Mathematics dependencies in the input data can be handled.

2. Theory and Algorithms

First we discuss the symmetric case. Let $[A] \in \mathbb{IR}^{n\times n}$ with $[a_{\mu\nu}] = [a_{\nu\mu}]$ for $\mu, \nu = 1, \ldots, n$. Then we call the set of matrices

$$\{A^{\text{sym}}\} := \{A \in \mathbb{R}^{n\times n} | A \in [A], A \text{ symmetric}\} \tag{2.1}$$

a *symmetric interval matrix*. $\{A^{\text{sym}}\} \notin \mathbb{IR}^{n\times n}$ is not an interval matrix in the usual sense.

$\{A^{\text{sym}}\} \subseteq [A]$ and $\{A^{\text{sym}}\} = [A]$ if and only if $\underline{a}_{\mu\nu} = \overline{a}_{\mu\nu}$ for $\mu, \nu = 1, \ldots, n, \mu \neq \nu$. A symmetric interval matrix $\{A^{\text{sym}}\}$ and a right hand side $[b] \in \mathbb{IR}^n$ define a family of linear systems

$$Ax = b, \qquad A \in \{A^{\text{sym}}\}, \qquad b \in [b] \tag{2.2}$$

with the corresponding solution set

$$\Sigma(\{A^{\text{sym}}\}, [b]) := \{x \in \mathbb{R}^n | Ax = b, A \in \{A^{\text{sym}}\}, b \in [b]\}. \tag{2.3}$$

Obviously $\Sigma(\{A^{\text{sym}}\}, [b]) \subseteq \Sigma([A], [b])$.

Theorem 1. *Let $\{A^{\text{sym}}\}$ be a symmetric interval matrix, $R \in \mathbb{R}^{n\times n}$, $\tilde{x} \in \mathbb{R}^n$ and $[b] \in \mathbb{IR}^n$.*

1) *Let $[z] \in \mathbb{IR}^n$ be defined by*

$$[z_i] := \sum_{\mu=1}^{n} r_{i\mu}([b_\mu] - [a_{\mu\mu}]\tilde{x}_\mu) - \sum_{\substack{\mu,\nu=1\\ \mu<\nu}}^{n} (r_{i\mu}\tilde{x}_\nu + r_{i\nu}\tilde{x}_\mu)[a_{\mu\nu}] \tag{2.4}$$

for $i = 1, \ldots, n$ then

$$[z] = \lozenge(\{R\cdot(b - A\tilde{x}) | A \in \{A^{\text{sym}}\}, b \in [b]\}). \tag{2.5}$$

2) *For $[w] \in \mathbb{IR}^n$ let $[x], [y] \in \mathbb{IR}^n$ be defined by*

$$[x] := [z] + (I - R[A])[w] \tag{2.6}$$

and

$$[y] := [\inf([z]) + \sup((I - R[A])[x]), \sup([z]) + \inf((I - R[A])[x])]. \tag{2.7}$$

If $[x] \subseteq \operatorname{int}([w])$ then R and all $A \in \{A^{\text{sym}}\}$ are non-singular and

$$\tilde{x} + [y] \subseteq \lozenge(\Sigma(\{A^{\text{sym}}\}, [b])) \subseteq \tilde{x} + [x]. \tag{2.8}$$

Remark. I denotes the $n \times n$ identity matrix and $\operatorname{int}([w])$ denotes the interior of $[w]$. In (2.7) *sup* and *inf* are to be understood componentwise and

$$\inf([z]) + \sup((I - R\cdot[A])[x]) \leq \sup([z]) + \inf((I - R\cdot[A])[x])$$

is assumed.

Proof.

1) Let $A \in \{A^{\text{sym}}\}$, $b \in [b]$ and r^i the ith row vector of R. Then the ith component of $R(b - A\tilde{x})$ satisfies

$$r^i(b - A\tilde{x}) = \sum_{\mu=1}^{n} r_{i\mu} b_\mu - \sum_{\mu=1}^{n} r_{i\mu}\left(\sum_{\nu=1}^{n} a_{\mu\nu}\tilde{x}_\nu\right)$$

$$= \sum_{\mu=1}^{n} r_{i\mu}(b_\mu - a_{\mu\mu}\tilde{x}_\mu) - \sum_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^{n} r_{i\mu} a_{\mu\nu}\tilde{x}_\nu$$

and $a_{\mu\nu} = a_{\nu\mu}$ yields

$$r^i(b - A\tilde{x}) = \sum_{\mu=1}^{n} r_{i\mu}(b_\mu - a_{\mu\mu}\tilde{x}_\mu) - \sum_{\substack{\mu,\nu=1 \\ \mu<\nu}}^{n} (r_{i\mu}\tilde{x}_\nu + r_{i\nu}\tilde{x}_\mu)a_{\mu\nu}. \tag{2.9}$$

By a Theorem of Moore ([14], page 23) we get

$$\{r^i(b - A\tilde{x}) | A \in \{A^{\text{sym}}\}, b \in [b]\} = [z_i], \qquad i = 1, \ldots, n$$

because in (2.4) each interval variable occurs only once and to the first power. Hence (2.5) is valid.

2) If $A, R \in \mathbb{R}^{n\times n}$, $\tilde{x}, b \in \mathbb{R}^n$, $[w] \in \mathbb{IR}^n$ and

$$R(b - A\tilde{x}) + (I - RA)\cdot[w] \subseteq \text{int}([w])$$

then from a Theorem of Rump ([18], page 59) it follows that R, A are non-singular and the exact solution $\hat{x}$ of $Ax = b$ is contained in $\tilde{x} + (R\cdot(b - A\tilde{x}) + (I - RA)\cdot[w])$.

Because of (2.6)

$$\bigcup \{R\cdot(b - A\tilde{x}) + (I - R[A])[w] | A \in \{A^{\text{sym}}\}, b \in [b]\} \subseteq [x]$$

and with $[x] \subseteq \text{int}([w])$ the application of the Theorem cited above yields the non-singularity of R and all $A \in \{A^{\text{sym}}\}$ and $\Diamond(\Sigma(\{A^{\text{sym}}\}, [b])) \subseteq \tilde{x} + [x]$.

To prove the last assertion we follow the theory developed in [20]. The identity

$$\tilde{x} + R(b - A\tilde{x}) = A^{-1}b - (I - RA)(A^{-1}b - \tilde{x})$$

yields

$$\begin{aligned}
&\tilde{x} + \{R(b - A\tilde{x}) | A \in \{A^{\text{sym}}\}, b \in [b]\} \\
&\quad\subseteq \Sigma(\{A^{\text{sym}}\}, [b]) - \{(I - RA)(A^{-1}b - \tilde{x}) | A \in \{A^{\text{sym}}\}, b \in [b]\} \\
&\quad\subseteq \Sigma(\{A^{\text{sym}}\}, [b]) - (I - R[A])(\Sigma(\{A^{\text{sym}}\}, [b]) - \tilde{x}) \\
&\quad\subseteq \Sigma(\{A^{\text{sym}}\}, [b]) - (I - R[A])[x].
\end{aligned}$$

By defining $Q := \{R\cdot(b - A\tilde{x}) | A \in \{A^{\text{sym}}\}, b \in [b]\}$, and $\Delta := (I - R[A])[x]$ from Lemma 1 in [20] it follows that

$$\inf(\Sigma(\{A^{\text{sym}}\}, [b])) \leq \tilde{x} + \inf(Q) + \sup(\Delta)$$

$$\sup(\Sigma(\{A^{\text{sym}}\}, [b])) \geq \tilde{x} + \sup(Q) + \inf(\Delta).$$

Taking the interval hull and observing that $[z] = \Diamond(Q)$ yields $\tilde{x} + [y] \subseteq \Diamond(\Sigma(\{A^{\text{sym}}\}, [b]))$. □

Now the following algorithm computes interval vectors $[x], [y] \in \mathbb{IR}^n$ and $\tilde{x} \in \mathbb{R}^n$ satisfying (2.8).

(1) Compute with some standard algorithm an approximation R of $m([A])$ and $\tilde{x} := R \cdot m([b])$. Optionally improve $\tilde{x}$ by *a* residual iteration.
(2) Calculate $[z]$ by formula (2.4); $[C] := I - R \cdot [A]$; $[x] := [z]$; $k := 0$;
(3) *repeat* $[w] := [x] \cdot [1 - \varepsilon, 1 + \varepsilon] + [-\mu, \mu]$; $k := k + 1$;
$[x] := [z] + [C] \cdot [w]$;
until $[x] \subseteq \operatorname{int}([w])$ *or* $k > 10$.
(4) *if* $[x] \subseteq \operatorname{int}([w])$ *then*
a) {all $A \in \{A^{\text{sym}}\}$ are non-singular and $\diamondsuit(\Sigma(\{A^{\text{sym}}\}, [b])) \subseteq \tilde{x} + [x]$}
b) {with $[y]$ by (2.7) holds $\tilde{x} + [y] \subseteq \diamondsuit(\Sigma(\{A^{\text{sym}}\}, [b]))$}
else {no inclusion can be computed}.

The above algorithm is a modification of Rump's algorithm ([18], page 62) for symmetric interval matrices. For a practical implementation on digital computers some remarks should be made. The operations in step (1) are the ordinary floating-point operations whereas the operations in step (2) to (4a) are the floating-point interval operations. The computation of $\tilde{x} + [y]$ must be implemented carefully. To calculate an interval vector contained in the interval hull $\diamondsuit(\Sigma(\{A^{\text{sym}}\}, [b]))$ obviously according to (2.7) we need to compute an inner estimation of $[z]$ and an outer estimation of $(I - R \cdot [A])[x]$. The outer estimation is calculated by using the ordinary floating-point interval operations. The inner estimation of $[z]$ can be computed by using the interval operations with inward roundings, i.e.

$$[\underline{a}, \bar{a}] * [\underline{b}, \bar{b}] := [\operatorname{Min}\{\Delta(\underline{a} * \underline{b}), \Delta(\underline{a} * \bar{b}), \Delta(\bar{a} * \underline{b}), \Delta(\bar{a} * \bar{b}),$$
$$\operatorname{Max}\{\nabla(\underline{a} * \underline{b}), \nabla(\underline{a} * \bar{b}), \nabla(\bar{a} * \underline{b}), \nabla(\bar{a} * \bar{b})]$$

where $* \in \{+, -, \cdot, /\}$ and Δ, ∇ denote the upward resp. downward rounding. The components of the vector $\mu \in \mathbb{R}^n$ in step (3) are equal to the smallest positive floating-point number and a good choice of $\varepsilon = 0.1$.

Now we discuss the case of skew-symmetric matrices. The $n \times n$ matrix A is called *skew-symmetric* if $A^t = -A$. Let $[A] \in \mathbb{IR}^{n \times n}$ with $[A]^t = -[A]$ and $[a_{ii}] = 0$ for $i = 1, \ldots, n$ then we call the set of matrices

$$\{A^{\text{skew}}\} := \{A \in \mathbb{R}^{n \times n} | A \in [A], A \text{ skew-symmetric}\} \tag{2.10}$$

a *skew-symmetric interval matrix*. A skew-symmetric interval matrix $\{A^{\text{skew}}\}$ and a right hand side $[b] \in \mathbb{IR}^n$ define a family of linear systems

$$Ax = b, \qquad A \in \{A^{\text{skew}}\}, \qquad b \in [b] \tag{2.11}$$

with the corresponding solution set

$$\Sigma(\{A^{\text{skew}}\}, [b]) := \{x \in \mathbb{R}^n | Ax = b, A \in \{A^{\text{skew}}\}, b \in [b]\}. \tag{2.12}$$

If $A \in \{A^{\text{skew}}\}$ then

$$r^i(b - A\tilde{x}) = \sum_{\mu=1}^{n} r_{i\mu} b_\mu - \sum_{\substack{\mu, \nu=1 \\ \mu < \nu}}^{n} (r_{i\mu}\tilde{x}_\nu - r_{i\nu}\tilde{x}_\mu) a_{\mu\nu}. \tag{2.13}$$

Obviously, by replacing in Theorem 1 formula (2.4) by

$$[z_i] = \sum_{\mu=1}^{n} r_{i\mu}[b_\mu] - \sum_{\substack{\mu,\nu=1\\ \mu<\nu}}^{n} (r_{i\mu}\tilde{x}_\nu - r_{i\nu}\tilde{x}_\mu)[a_{\mu\nu}] \tag{2.14}$$

and symbol A^{sym} by A^{skew} we analogously get a theorem and an algorithm for the skew-symmetric case.

In many applications dependencies in the right hand side occur. This is for example the case for many models in Operations Research (product mix problems, blending problems, investment problems,...). To handle this case let $(J_k)_{k=1}^{l}$ be a partitioning of the index set $\{1,\dots,n\}$, i.e.

$$J_k \subseteq \{1,\dots,n\}, \quad J_{k_1} \cap J_{k_2} = \emptyset \quad \text{for } k_1 \neq k_2, \quad \bigcup_{k=1}^{l} J_k = \{1,\dots,n\}.$$

Let $[\beta_k] \in \mathbb{IR}^n$ for $k = 1, \dots, l$ and $s \in \mathbb{R}^n$ then we call the set

$$\{b^{\text{dep}}\} := \{b \in \mathbb{R}^n | b_j = s_j\beta_k, j = 1,\dots,n \text{ with } \beta_k \in [\beta_k] \text{ for } j \in J_k\} \tag{2.15}$$

a *dependent interval vector* with respect to the *partitioning* $(J_k)_{k=1}^{l}$ and the *multipliers* $s \in \mathbb{R}^n$. $\{b^{\text{dep}}\}$ is not an interval vector in the usual sense.

A dependent right hand side $\{b^{\text{dep}}\}$ and an interval matrix $[A]$ define a family of linear systems

$$Ax = b, A \in [A], b \in \{b^{\text{dep}}\} \tag{2.16}$$

with the corresponding solution set

$$\Sigma([A], \{b^{\text{dep}}\}) := \{x \in \mathbb{R}^n | Ax = b, A \in [A], b \in \{b^{\text{dep}}\}\} \tag{2.17}$$

Obviously $\Sigma([A], \{b^{\text{dep}}\}) \subseteq \Sigma([A], [b])$ if $[b]$ is defined by $[b_j] = s_j[\beta_k]$ for all $j \in J_k$, $k = 1, \dots, l$.

Theorem 2. *Let $\{b^{\text{dep}}\}$ be a dependent interval vector with respect to the partitioning $(J_k)_{k=1}^{l}$ and the multipliers $s \in \mathbb{R}^n$, $[A] \in \mathbb{IR}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$, $\tilde{x} \in \mathbb{R}^n$, $[b] \in \mathbb{IR}^n$.*

1) *Let $[z] \in \mathbb{IR}^n$ be defined by*

$$[z_i] := \sum_{k=1}^{l} \left(\sum_{\mu \in J_k} r_{i\mu}s_\mu \right)[\beta_k] - r^i([A]\tilde{x}) \tag{2.18}$$

for $i = 1, \dots, n$. Then

$$[z] = \lozenge(\{R(b - A\tilde{x}) | A \in [A], b \in \{b^{\text{dep}}\}\})$$

2) *For $[w] \in \mathbb{IR}^n$ define $[x]$, $[y] \in \mathbb{IR}^n$ by* (2.6) *and* (2.7). *If $[x] \subseteq \text{int}([w])$ then R and all $A \in [A]$ are non-singular and*

$$\tilde{x} + [y] \subseteq \lozenge(\Sigma([A], \{b^{\text{dep}}\})) \subseteq \tilde{x} + [x] \tag{2.19}$$

Proof. If $b \in \{b^{\text{dep}}\}$ then

$$r^i(b - A\tilde{x}) = \sum_{\mu=1}^{n} r_{i\mu} b_\mu - r^i(A\tilde{x})$$
$$= \sum_{k=1}^{l} \left(\sum_{\mu \in J_k} r_{i\mu} s_\mu \right) \beta_k - r^i(A\tilde{x}).$$

Obviously the proof of theorem 3 can be finished analogously to the proof of theorem 1. □

The algorithm described for symmetric interval matrices works in this case if $[z]$ is defined by formula (2.18).

3. Numerical Examples

We present two examples to illustrate our previous analysis. In the examples we use a machine with base 16 and machine unit eps $:= 16^{-13} = 0.22\ldots 10^{-16}$. The algorithms are implemented by the programming language CALCULUS [19], an interactive programming environment supporting Kulisch's arithmetic [12], the algorithms of the ACRITH subroutine library [9], [10] and LINPACK, EISPACK. In the following we display the results of the above algorithms with five significant figures.

In our first example the (symmetric) Hilbert matrix

$$(H_n)_{ij} := (\text{lcm}(1, \ldots, 2n - 1))/(i + j + 1)$$

of dimension $n = 5$ is considered (lcm: least common multiple of all denominators). The right hand side is given by $[b] = H_5 \cdot x$ with

$$x = (1.00000, -0.50000, 0.37500, -0.31250, 0.27344).$$

The coefficients of $[A]$ are defined by

$$[a_{i,i+1}] = [a_{i+1,i}] := [H_{i,i+1} \cdot (1 - 3 \cdot 10^{-6}), H_{i,i+1} \cdot (1 + 3 \cdot 10^{-6})], \quad i = 1, \ldots, n - 1$$

and $[a_{ij}] = (H_5)_{ij}$ for the other coefficients. In table 3.1 the inner and outer bounds $\tilde{x} + [y], \tilde{x} + [x]$ of the symmetric solution set $\Sigma(\{A^{\text{sym}}\}, [b])$ are displayed whereas in table 3.2 the inclusion vector $[u]$ of $\Sigma([A], [b])$ (calculated by the routine DILIN of ACRITH) and the diameters $d([u_i])/d((\tilde{x} + [x])_i)$ for $i = 1, \ldots, 5$ are displayed

Table 3.1. Inner and outer bounds for $\Sigma(\{A^{\text{sym}}\}, [b])$

$\tilde{x} + [x]$	$\tilde{x} + [y]$
[0.99305, 1.00695]	[0.99411, 1.00589]
[−0.50228, −0.49772]	[−0.50147, −0.49854]
[0.37398, 0.37602]	[0.37465, 0.37534]
[−0.31385, −0.31115]	[−0.31329, −0.31171]
[0.27168, 0.27520]	[0.27219, 0.27469]

Table 3.2. Inclusion vector $[u]$ of $\Sigma([A],[b])$ and ratio of diameters

$[u]$	$d([u])/d(\tilde{x}+[x])$
[0.83461, 1.16539]	23.8
[−0.62788, −0.37212]	55.9
[0.27037, 0.47963]	102.6
[−0.40115, −0.22385]	65.3
[0.19647, 0.35041]	43.9

These results show that a severe overestimation may occur if the input data of a symmetric interval matrix are handled independently. For example the diameter of the third component $d([u_3])$ overestimates the diameter of the third component of the symmetric solution set by a factor of about 100. Moreover, the algorithm described in section 2 for the symmetric case delivers very sharp inner and outer bounds of the symmetric solution set $\Sigma(\{A^{\mathrm{sym}}\},[b])$.

In the second example the matrix

$$A = \begin{bmatrix} -1.271e0 & 6.713e-1 & -3.095e-1 & 6.190e-1 \\ -1.312e0 & 6.875e-1 & -3.125e-1 & 6.250e-1 \\ -9.688e-1 & 3.124e-2 & 3.126e-2 & 9.375e-1 \\ -9.496e-1 & 2.157e-2 & 3.119e-2 & 9.376e-1 \end{bmatrix}$$

is considered. This matrix is well-conditioned with an approximative l_1 condition number $2.504\cdot 10^3$. We define an interval matrix $[A]$ by

$$[a_{ij}] := [a_{ij} - 10^{-5}\cdot|a_{ij}|, a_{ij} + 10^{-5}\cdot|a_{ij}|], \qquad i,j = 1,\ldots,n$$

and a dependent right hand side with partitioning and multipliers

$$J_1 := \{1,2\}, \qquad J_2 := \{3,4\}, \qquad s := (1,1,1,1)^t$$

and

$$[\beta_1] = [\beta_2] := [1 - 5\cdot 10^{-3}, 1 + 5\cdot 10^{-3}].$$

In Table 3.3 the inner and outer bounds $\tilde{x}+[y]$, $\tilde{x}+[x]$ of the solution set $\Sigma([A],\{b^{\mathrm{dep}}\})$ are displayed.

In Table 3.4 the inclusion vector $[u]$ of $\Sigma([A],[b])$ (calculated by the routine DILIN in ACRITH) and the proportion of the diameters of $[u]$ and $\tilde{x}+[x]$ are given. The right hand side $[b]$ is defined by $[b_i] = [\beta_1]$ for $i = 1,\ldots,4$.

Table 3.3. Inner and outer bounds for $\Sigma([A],\{b^{\mathrm{dep}}\})$

$\tilde{x}+[x]$	$\tilde{x}+[y]$
[1.0199, 1.0621]	[1.0206, 1.0614]
[2.0322, 2.1283]	[2.0337, 2.1268]
[1.0220, 1.1382]	[1.0237, 1.1365]
[2.0132, 2.0610]	[2.0139, 2.0604]

Table 3.4. Inclusion vector $[u]$ of $\Sigma([A],[b])$ and ratio of diameters

$[u]$	$d([u])/d(\tilde{x}+[x])$
$[-1.1091\quad, 3.1911]$	102.16
$[-3.2125\quad, 7.3730]$	110.23
$[-5.2343\quad, 7.3945]$	108.78
$[\ 1.9678e-1, 3.8775]$	77.962

In the case of dependencies in the right hand side a severe overestimation occurs. Taking into consideration dependencies of the right hand side very sharp bounds of the corresponding solution set are calculated whereas without regarding those dependencies even the signs of the solution set $\Sigma([A],[b])$ cannot be determined and the ratio of the diameters varies between 78 and 110.

Acknowledgement

I wish to thank Prof. Dr. S. M. Rump for many valuable and helpful discussions.

References

[1] Alefeld, G., Herzberger, J.: Einführung in die Intervallrechnung, B. I. Wissenschaftsverlag 1974.
[2] Bauch, H., Jahn, K.-U., Oelschlägel, D., Süsse, H., Wiebigke, V.: Intervallmathematik, Theorie und Anwendungen, Mathematische-Naturwissenschaftliche Bibliothek, Band 72, B. G. Teubner 1987.
[3] Bleher, J. H., Rump, S. M., Kulisch, U. W., Metzger, M., Ullrich, Ch., Walter, W.: FORTRAN-SC: A Study of a FORTRAN Extension for Engineering/Scientific Computation with Access to ACRITH, Computing *39*, 93–110, Springer (1987).
[4] Hansen, E.; Smith, R.: Interval arithmetic in matrix computations, Part II, SIAM J. Numer. Anal., *4*, 1–9 (1967).
[5] Hansen, E.: On the solution of linear algebraic equations with interval coefficients, Linear Algebra Appl. *2*, 153–165 1969.
[6] Husung, D.: —ABACUS—Programmierwerkzeug mit hochgenauer Arithmetik für Algorithmen mit verfizierten Ergebnissen, Diplomarbeit, Karlsruhe 1988.
[7] Husung, D.: Precompiler for Scientific Computation (TPX), Informatik III, TU Hamburg-Harburg 1989.
[8] Husung, D., Rump, S. M.: ABACUS, Proceedings SCAN '89, Tagung "Wissenschaftliches Rechnen und Programmiersprachen", Basel 1989.
[9] IBM High-Accuracy Arithmetic Subroutine Library (ACRITH), General Information Manual, GC 33-6163-02, 3rd ed. 1986.
[10] IBM High-Accuracy Arithmetic Subroutine Library (ACRITH), Program Description and User's Guide, SC 33-6164-02, 3rd ed. 1986.
[11] Krawczyk, R.: Newton-Algorithmen zur Bestimmung von Nullstellen mit Fehlerschranken, Computing, *4* 187–201, 1969.
[12] Kulisch, U. W.: Computer arithmetic in theory and practice, New York: Academic Press 1981.
[13] Kulisch U. W. (ed.): PASCAL-SC: A PASCAL extension for scientific computation, information manual and floppy disks, Version IBM PC, Stuttgart, B. G. Teubner; Chichester: John Wiley & Sons 1987.
[14] Moore, R. E.: Methods and applications of interval analysis, SIAM Philadelphia 1979.
[15] Neumaier, A.: Interval methods for systems of equations, Cambridge University Press, to be published.
[16] Rohn, J.: Solving interval linear systems, Freiburger Intervallberichte *84/7*, 1–30 (1984).

[17] Rump, S. M.: Kleine Fehlerschranken bei Matrixproblemen, Dr.-Dissertation, Institut für Angewandte Mathematik, Universität Karlsruhe 1980.
[18] Rump, S. M.: Solving algebraic problems with high accuracy, Habilitationsschrift, in: A New Approach to Scientific Computation, Hrsg. U. W. Kulisch und W. L. Miranker, ACADEMIC PRESS, 51–120, 1983.
[19] Rump, S. M.: CALCULUS, in: "Wissenschaftliches Rechnen mit Ergebnisverifikation", Kulisch, U. (ed.) Berlin: Vieweg und Akademie Verlag 1989.
[20] Rump, S. M.: Rigorous sensitivity analysis for systems of linear and nonlinear equations, MATH. of COMP., *54*, *190*, 721–736 (1990).
[21] Nenmaier, A.: Overestimation in linear interval equations, SIAM J. Num. Anal., *24*, 207–214 (1987).
[22] Nenmaier, A.: Rigorous sensitivity analysis for parameter-dependent systems of equations, (to appear).

Dr. C. Jansson
Technische Informatik III
Technische Universität
Eißendorfer Straße 38
D-W-2100 Hamburg 90
Federal Republic of Germany

Verleger: Springer-Verlag KG, Sachsenplatz 4–6, A-1201 Wien. – Herausgeber: Prof. Dr. Hans J. Stetter, Institut für Angewandte und Numerische Mathematik der Technischen Universität Wien, Wiedner Hauptstraße 6–10, A-1040 Wien. – Redaktion: Wiedner Hauptstraße 6–10, A-1040 Wien. – Satz und Umbruch: Asco Trade Typesetting Limited, Hong Kong; Reproduktion und Offsetdruck: Novographic, Ing. W. Schmid, Maurer-Lange-Gasse 64, A-1238 Wien. – Verlagsort: Wien. – Herstellungsort: Wien. – Printed in Austria.