Topics in Interval Analysis

EDITED BY

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OXFORD
AT THE CLARENDON PRESS
1969

Oxford University Press, Ely House, London W. 1

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PRINTED IN GREAT BRITAIN

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1. Introduction

In this chapter we show how interval arithmetic can be used to bound the solution to certain two-point boundary-value problems for ordinary differential equations. Our method can be applied to non-linear equations but in some such cases we assume initial crude bounds to be given. However, for the linear case, no initial bounds are required.

To simplify the presentation, we consider only a single equation of second order. The method to be discussed can be easily extended to the more general case. In [6] (p. 83), Moore indicates that the problem

$$y'' = f(x,y), \qquad y(a) = y_0, \qquad y(h) = y_1$$

can be solved, with strict bounds on the error, by formulating the problem as an integral equation and using the method of Chapter 9 of [6]. See also Chapter 6 of this book. In the following, we present an alternative procedure for solving the more general equation

$$y'' = f(x,y,y') \tag{1.1}$$

with boundary conditions

$$g_1[a,y(a),y'(a)] = 0, \qquad g_2[b,y(b),y'(b)] = 0. \tag{1.2}$$

For brevity, we replace equations (1.2) by the special simple conditions

$$y(a) = y_0, \qquad y(b) = y_n. \tag{1.3}$$

However, use of (1.2) instead of (1.3) introduces no difficulties. Our method is essentially an interval analytic extension of the difference approximation method in common use (see, for example, [2]). The modification of our method to use (1.2) instead of (1.3) follows the same steps as those of section 12 of Chapter 4 of [2].

We assume the problem expressed by (1.1) and (1.2) has a unique solution, bounded in $[a, b]$, and that its first four derivatives are continuous and bounded in $[a, b]$. We impose further conditions at the end of section 3.

We also assume f is a rational function of x, y, and y'. If this is not the case, it may be possible to obtain a system of differential equations entailing only rational functions. For details of such a step, see, for example, section 11.2 of [6]. This assumption is not necessary, in general, since there are means for computing intervals containing the 'value' of irrational functions with interval arguments. For example, see Chapter 1 by Moore. We consider some irrational examples in sections 10 and 11.

2. The basic step of the method

Divide the interval $[a, b]$ into sub-intervals

$$X_i = [x_i, x_{i+1}] \quad (i = 0, 1, \ldots, n-1)$$

where $x_0 = a$ and $x_n = b$. The meshpoints x_i need not be equally spaced although we assume them to be. At each interior meshpoint $x_1, \ldots, x_{n-1}$, we write discrete approximations for the derivatives in the differential equation (1.1). The error in these approximations can be analytically expressed and then bounded by use of interval arithmetic. The simplest central difference approximations are the well-known formulae

$$y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1}) - \frac{h^2}{6} y'''(\xi_i) \quad (i = 1, \ldots, n-1) \tag{2.1}$$

and
$$y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i - y_{i-1}) - \frac{h^2}{12} y^{(4)}(\eta_i) \quad (i = 1, \ldots, n-1), \tag{2.2}$$

where $h = x_{i+1} - x_i$ and y_i denotes $y(x_i)$, etc. The quantities ξ_i and η_i are unknown except that $\xi_i \in X_i^*$ and $\eta_i \in X_i^*$ where

$$X_i^* = X_i \cup X_{i-1} = [x_{i-1}, x_{i+1}].$$

We later show how to bound the error terms. For now, assume we know intervals A_i and B_i such that $y'''(\xi_i) \in A_i$ and $y^{(4)}(\eta_i) \in B_i$ for ξ_i and η_i in X_i^*. Substituting these bounding intervals for the respective quantities in (2.1) and (2.2) and substituting the results into (1.1) (with $x = x_i$), we obtain

$$\frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) - \frac{h^2}{12} B_i$$
$$= f\left[x_i, y_i, \frac{1}{2h}(y_{i+1} - y_{i-1}) - \frac{h^2}{6} A_i\right] \quad (i = 1, \ldots, n-1). \tag{2.3}$$

Since y_0 and y_n are given, we thus have $n-1$ equations in the $n-1$ unknowns $y_1,\ldots, y_{n-1}$. If the equations are linear, we can solve them by (say) the interval arithmetic method recommended in [5] (for which a computer program is given in [7]). If the equations are non-linear, a method in [4] or [6] can be used. We thus obtain an interval y_i^I containing y_i ($i = 1,\ldots, n-1$).

3. Obtaining A_i and B_i

We now consider how the bounds A_i and B_i can be obtained. To do this, we assume we have bounds on y and y'; that is, assume we know $y(x) \in Y_i$ and $y'(x) \in Y'_i$ for $x \in X_i$. In later sections we discuss how to find the intervals Y_i and Y'_i.

Differentiating equation (1.1), we have

$$y''' = f_x + y'f_y + y''f_{y'}, \tag{3.1}$$

where $f_y \equiv \partial f/\partial y$ and $f_{y'} \equiv \partial f/\partial y'$. Substituting for y'' in (3.1) from (1.1), we obtain

$$y''' = p(x,y,y'), \tag{3.2}$$

where

$$p(x,y,y') \equiv f_x + y'f_y + ff_{y'} \tag{3.3}$$

is a function of x, y, and y' alone.

Similarly we find

$$y^{(4)} = q(x,y,y') \tag{3.4}$$

by differentiating (3.2) and substituting for y'' as before.

We can bound y''' over an interval X_i by evaluating $p(X_i, Y_i, Y'_i)$ using interval arithmetic. Denote

$$A_i = p(X_i, Y_i, Y'_i) \cup p(X_{i-1}, Y_{i-1}, Y'_{i-1}). \tag{3.5}$$

Similarly,

$$B_i = q(X_i, Y_i, Y'_i) \cup q(X_{i-1}, Y_{i-1}, Y'_{i-1}). \tag{3.6}$$

The intervals A_i and B_i are the quantities required in (2.3).

We assume that $p(X_i, Y_i, Y'_i)$ and $q(X_i, Y_i, Y'_i)$ are bounded for all $i = 0,\ldots, n-1$. This rules out many interesting differential equations. For example, we cannot solve $y'' = y/x$ if $0 \in [a,b]$.

4. Improving Y'_i

We assume the bounding intervals Y_i and Y'_i were initially crude. We now consider how to improve these bounds.

Using Taylor series with remainder we easily find

$$y'(x) = \frac{1}{h}(y_{i+1}-y_i) + \frac{1}{2h}[(x_{i+1}-x)^2 y''(\theta_i) - (x-x_i)^2 y''(\phi_i)] \tag{4.1}$$

for any point $x \in X_i$, where $\theta_i \in X_i$ and $\phi_i \in X_i$. Denote

$$Y_i'' = f(X_i, Y_i, Y_i'). \tag{4.2}$$

From (1.1), $y''(\theta_i) \in Y_i''$ and $y''(\phi_i) \in Y_i''$ and hence from (4.1),

$$\begin{aligned} y'(x) &\in \frac{1}{h}(y_{i+1}^I - y_i^I) + \frac{1}{2h}\{(x_{i+1}-X_i)^2 Y_i'' - (X_i - x_i)^2 Y_i''\} \\ &= \frac{1}{h}(y_{i+1}^I - y_i^I) + \frac{h}{2}\{[0,1]^2 Y_i'' - [0,1]^2 Y_i''\} \\ &= \frac{1}{h}(y_{i+1}^I - y_i^I) + \frac{h}{2} w\{[0,1]Y_i''\}[-1,1] \end{aligned}$$

for any $x \in X_i$, where $w\{[0,1]Y_i''\}$ denotes the width of the interval $[0,1]Y_i''$. Denote

$$Y_i' = \frac{1}{h}(y_{i+1}^I - y_i^I) + \frac{h}{2} w([0,1]Y_i'')[-1,1]. \tag{4.3}$$

Then $y'(x) \in Y_i'$ for any $x \in X_i$ and Y_i' is the (improved, in general) bound we sought. We use the same notation Y_i' for the old crude bound and the new improved bound on $y'(x)$ for $x \in X_i$. At any stage of our method, Y_i' denotes the current best approximation. In practice we should use the intersection of the old and new intervals.

5. Improving Y_i

We now use the improved bound Y_i' to improve Y_i. Using Taylor series, we easily find

$$y(x) = \tfrac{1}{2}[y_i + y_{i+1} + (x_{i+1}-x)y'(\mu_i) - (x - x_i)y'(\nu_i)] \tag{5.1}$$

for any $x \in X_i$, where $\mu_i \in X_i$ and $\nu_i \in X_i$. Since $y'(\mu_i) \in Y_i'$ and $y'(\nu_i) \in Y_i'$, we have

$$\begin{aligned} y(x) &\in \tfrac{1}{2}[y_i^I + y_{i+1}^I + (x_{i+1}-X_i)Y_i' - (X_i - x_i)Y_i'] \\ &= \tfrac{1}{2}(y_i^I + y_{i+1}^I) + \frac{h}{2}([0,1]Y_i' - [0,1]Y_i') \end{aligned}$$

for any $x \in X_i$. Denote

$$Y_i = \tfrac{1}{2}(y_i^I + y_{i+1}^I) + \frac{h}{2} w([0,1]Y_i')[-1,1]. \tag{5.2}$$

Then $y(x) \in Y_i$ for $x \in X_i$. This new value of Y_i replaces the original crude value; and, as before for Y_i', the intersection of the two can be used.

6. The iterative method

We are now able to describe the method we propose. We proceed in the following steps.

(*a*) Procure crude bounds Y_i and Y_i' for $i = 0, \ldots, n-1$ (see sections 8–11).

(*b*) Evaluate A_i using (3.5) and B_i using (3.6) for $i = 1,\ldots, n-1$.
(*c*) Using (1.3), solve equations (2.3) for y_i^I $(i = 1,\ldots, n-1)$.
(*d*) Find improved bounds Y_i' $(i = 0,\ldots, n-1)$ using (4.3).
(*e*) Find improved bounds Y_i $(i = 0,\ldots, n-1)$ using (5.2).
(*f*) Iterate steps (*b*)–(*e*).

The iteration can proceed either until the error bounds are sufficiently sharp or until successive iterates differ by a sufficiently small amount. Note that for fixed finite precision arithmetic, a stage will be reached where no improvement occurs.

We have assumed convergence. If this does not occur, the fact is almost immediately revealed. In theory, the likelihood of convergence is enhanced by reducing h. In practice, this may not help because the number of interval equations (2.3) increases and may be difficult to solve sharply.

We wish next to present a general procedure for obtaining crude bounds when the differential equation is linear. To do this, we first require some preliminary concepts which we now consider.

7. Computation with variable intervals

Let M_r and N_r be intervals whose end-points are specifically given numbers. Let W be an unspecified variable interval. We cannot unambiguously express $M_r W$ explicitly in terms of the end-points of M_r and W since the end-points of $M_r W$ depend upon the unknown signs and magnitudes of W. To compute with variable intervals, we can represent them as $N_r W$. Then $M_r(N_r W)$ can be 'computed' by evaluating $N_{r+1} = M_r N_r$ and representing the result in the form $N_{r+1} W$.

If we assume W is symmetric about the origin so that $W = [-w, w]$, we can simplify the arithmetic. We can then replace $N_r = [n_r^L, n_r^R]$ by a positive real number n_r since

$$N_r[-w, w] = n_r[-w, w],$$

where $n_r = |N_r| = \max(|n_r^L|, |n_r^R|)$. Similarly

$$W/M_r = W|1/M_r|. \tag{7.1}$$

In the next section, we take advantage of this simplification.

8. Obtaining Y_i and Y_i' in the linear case

If boundary conditions of the form (3) are given, we choose to seek bounds of the form

$$Y_i = \{y(a)+y(b)\}/2+U \tag{8.1}$$

and

$$Y_i' = \frac{y(b)-y(a)}{b-a}+V, \tag{8.2}$$

where $U = [-u, u]$ and $V = [-v, v]$ so that U and V are symmetric about the origin. If the boundary conditions are not of the form (3), we can simply choose $Y_i = U$ and $Y'_i = V$.

Using the analytically expressed bounds (8.1) and (8.2), we perform steps (*b*)–(*e*) of our method described in section 6. We use the arithmetic described in section 7.

Let $\overline{Y}_i$ and $\overline{Y}'_i$ denote the new bounds on the solution and its derivative, obtained in this way. We find

$$\overline{Y}_i = M_i + c_i U + d_i V \tag{8.3}$$

and

$$\overline{Y}'_i = M'_i + c'_i U + d'_i V \tag{8.4}$$

for $i = 0,\ldots, n-1$, where $c_i \geqslant 0$, $d_i \geqslant 0$, $c'_i \geqslant 0$, and $d'_i \geqslant 0$. Denote $M_i = [p_i, q_i]$, $M'_i = [p'_i, q'_i]$, $\overline{Y}_i = [\bar{y}_i^L, \bar{y}_i^R]$, and $\overline{Y}'_i = [\bar{z}_i^L, \bar{z}_i^R]$. From (8.3) and (8.4),

$$\begin{aligned} \bar{y}_i^L &= p_i - c_i u - d_i v, \\ \bar{y}_i^R &= q_i + c_i u + d_i v, \\ \bar{z}_i^L &= p'_i - c'_i u - d'_i v, \\ \bar{z}_i^R &= q'_i + c'_i u + d'_i v, \end{aligned} \tag{8.5}$$

for $i = 0,\ldots, n-1$.

Let $r = \{y(a)+y(b)\}/2$ and $s = \{y(b)-y(a)\}/(b-a)$. Then from (8.1) and (8.5), the differences between the old and new left end-points of the bounds on $y(x)$ in X_i are

$$\begin{aligned} \Delta y_i^L &= (p_i - c_i u - d_i v) - (r - u) \\ &= p_i - r + (1-c_i)u - d_i v \quad (i = 0,\ldots, n-1) \end{aligned} \tag{8.6}$$

and the changes in the right end-points are

$$\begin{aligned} \Delta y_i^R &= (q_i + c_i u + d_i v) - (r + u) \\ &= q_i - r - (1-c_i)u + d_i v \quad (i = 0,\ldots, n-1). \end{aligned} \tag{8.7}$$

Similarly, the changes in the end-points of the interval containing $y'(x)$ in X_i are

$$\Delta z_i^L = p'_i - s - c'_i u + (1-d'_i)v \tag{8.8}$$

and

$$\Delta z_i^R = q'_i - s + c'_i u - (1-d'_i)v \tag{8.9}$$

for $i = 0,\ldots, n-1$.

The new bounds are strict improvements over the old if

$$\Delta y_i^L > 0, \quad \Delta y_i^R < 0, \quad \Delta z_i^L > 0, \quad \Delta z_i^R < 0 \quad (i = 0,\ldots, n-1). \tag{8.10}$$

We now argue that for h sufficiently small, the inequalities (8.10) will, in fact, hold provided the initial bounds are not already too sharp.

We performed steps (*b*)–(*e*) of the procedure in section 6. Each of these steps involved use of a formula in which the error term was multiplied by a positive integer power of h. Hence the non-negative numbers c_i, d_i, c'_i, and d'_i are all $O(h)$. We assume h is so small that

$$c_i < 1, \qquad d'_i < 1 \quad (i = 1,\ldots, n-1). \tag{8.11}$$

From (8.11), we see that the coefficients $1-c_i$ in (8.6) and (8.7) and $1-d'_i$ in (8.8) and (8.9) are positive. This (along with another condition given later) enables us to satisfy conditions (8.10) by choosing u and v to be related in an appropriate way.

Since we seek only crude bounds, we shall not attempt to obtain a best result but shall sacrifice sharpness (in the crude bounds) for simplicity of method. We do not, of course, drop the requirement that our bounds be strict.

Using (8.6)–(8.9) and noting (8.11), we rewrite (8.10) as

$$0 < (p_i-r)/d_i+u(1-c_i)/d_i-v, \tag{8.12}$$

$$0 < (r-q_i)/d_i+u(1-c_i)/d_i-v, \tag{8.13}$$

$$0 < (p'_i-s)/(1-d'_i)-uc'_i/(1-d'_i)+v, \tag{8.14}$$

$$0 < (s-q'_i)/(1-d'_i)-uc'_i/(1-d'_i)+v, \tag{8.15}$$

for $i = 0,\ldots, n-1$. We assume $d_i > 0$ although the case $d_i = 0$ causes no difficulty. However, $d_i = 0$ only if f is independent of y. For simplicity, we omit discussion of such cases. Define

$$\alpha_1 = \min\{(p_i-r)/d_i\}, \quad \alpha_2 = \min\{(r-q_i)/d_i\},$$
$$\alpha'_1 = \min\{(p'_i-s)/(1-d'_i)\}, \quad \alpha'_2 = \min\{(s-q'_i)/(1-d'_i)\},$$
$$\beta = \min\{(1-c_i)/d_i\}, \quad \text{and} \quad \beta' = \max\{c'_i/(1-d'_i)\},$$

where the max and min are taken over all $i = 0,\ldots, n-1$. Define $\alpha = \min(\alpha_1, \alpha_2)$ and $\alpha' = \min(\alpha'_1, \alpha'_2)$. Then (8.12) and (8.13) are satisfied if $u \geqslant 0$ and $v \geqslant 0$ are such that

$$0 < \alpha+\beta u-v \tag{8.16}$$

and (8.14) and (8.15) are satisfied if

$$0 < \alpha'-\beta' u+v. \tag{8.17}$$

The right members of (8.16) and (8.17) can be viewed as lines in the (u, v) plane. Relation (8.16) says the point (u, v) must lie below the line

$$\alpha+\beta u-v = 0 \tag{8.18}$$

and (8.17) says (u, v) must lie above the line

$$\alpha' - \beta' u + v = 0. \tag{8.19}$$

It is easily seen that $0 < \beta = O(1/h)$ and $0 < \beta' = O(1)$. Hence for h sufficiently small,

$$\beta' < \beta \tag{8.20}$$

and there exist points (u, v) satisfying both (8.16) and (8.17). Hereafter, we assume (8.20) to hold.

In general, conditions (8.11) and (8.20) can be satisfied by choosing h sufficiently small. However, this is not always the case. If equations (2.3) are nearly linearly dependent, y_i^I may be large in magnitude. Subsequently, terms which are supposedly relatively small may not be so and our method can fail. Note that for the eigenvalue problem wherein equations (2.3) are, in fact, linearly dependent, our method fails completely.

We choose (u, v) to lie on the line whose slope is the average of the slopes of the lines (8.18) and (8.19) and which passes through their point of intersection. That is, we choose (u, v) to lie on the line

$$\alpha - \alpha' + (\beta + \beta')u - 2v = 0. \tag{8.21}$$

A point on this line satisfies (8.16) and (8.17) if $u = u_0 + \Delta u$ for all $\Delta u > 0$ where

$$u_0 = -(\alpha + \alpha')/(\beta - \beta'). \tag{8.22}$$

It is easily seen that $u_0 = O(1)$. Note we also require $u > 0$.

Substituting for v from (8.21) into (8.6) and (8.7) and using the definitions of α, α', β, and β', we find

$$\Delta y_i^L \geqslant \frac{d_i}{2}(\beta - \beta')(u - u_0) \tag{8.23}$$

and

$$\Delta y_i^R \leqslant -\frac{d_i}{2}(\beta - \beta')(u - u_0) \tag{8.24}$$

respectively. Whatever value of $u > u_0$ we use in (8.1), we now see that we could replace it by

$$u^* = u - \frac{d}{2}(\beta - \beta')(u - u_0),$$

where $d = \min(d_i)$ for $i = 0, \ldots, n-1$. That is, we could reduce $u - u_0$ by

$$\Delta(u - u_0) = \left\{1 - \frac{d}{2}(\beta - \beta')\right\}(u - u_0). \tag{8.25}$$

Now $1 - (\beta - \beta')d/2 \geqslant (1 + c_i + d\beta')/2 > 0$ and hence $u - u_0$ can always be reduced by a positive fixed fraction of itself if $u > u_0$.

Similarly, from (8.8) and (8.9) we find that $v-v_0$ can be reduced by an amount

$$\Delta(v-v_0) = \frac{\beta-\beta'}{\beta+\beta'}(1-d')(v-v_0), \tag{8.26}$$

where $d' = \max(d'_i)$ for $i = 0,..., n-1$ and

$$v_0 = -\frac{\alpha\beta'+\beta\alpha'}{\beta-\beta'}.$$

Hence both $u-u_0$ and $v-v_0$ can be reduced by fixed positive fractions of themselves if $u > u_0$ and $v > v_0$. We have assumed y and y' bounded for all $x \in [a, b]$. Suppose we choose u and v satisfying (8.21) and so large that (8.1) and (8.2) do, in fact, bound y and y' in $[a, b]$. If

$$u > u_1 = \max(0, u_0) \quad \text{and} \quad v > v_1 = \max(0, v_0),$$

we can reduce both u and v, keeping (8.21) satisfied, until as the limit of an infinite sequence of steps, $u = u_1$ and $v = v_1$. (Actually we will find $u \leqslant u_1$ and $v \leqslant v_1$ since we have used bounds rather than true values of Δy_i^L, etc.) Thus, letting $u = u_1$ in (8.1) and $v = v_1$ in (8.2) yields actual bounds on y and y' for $x \in [a, b]$.

It is quite easy to obtain results which are slightly sharper, in general. Replace $<$ by $\leqslant$ in (8.12)–(8.15) and substitute for v in terms of u from (8.21). Find the smallest value of u satisfying all these relations for all $i = 0,..., n-1$. This value, substituted into (8.1), yields bounds on y. Substituting this value of u into (8.21) and solving for v yields a value which when used in (8.2) yields bounds on y'.

In general, still better results can be obtained by solving a linear programming problem. We minimize $\phi(u, v) = u$ subject to the constraints (8.10). The values of u and v for the solution point (u, v) yield bounds as before.

Proof of the validity of the statements in the last two paragraphs can be obtained in the manner used to prove $u = u_1$ and $v = v_1$ provide actual bounds.

In practice, a relatively large value of h could be used in obtaining the crude error bounds. If the bounding procedure cannot be completed because (8.11) or (8.20) fails to hold, then h can be reduced and the process repeated. However, the crude bounding procedure is quite simple to apply. Moreover, it yields sharper results for moderately small h. Hence there is no great advantage in using large h.

9. Example

We illustrate the above analysis with an example considered by Collatz on pp. 178 and 179 of [1]. Consider

$$y'' = 2x^{-2}y - 1/x \tag{9.1}$$

with boundary conditions

$$y(2) = y(3) = 0. \tag{9.2}$$

Following Collatz, we let $h = 1/3$. In general practice, however, it is necessary to choose a machine representable value of h.

Differentiating (9.1), we find

$$y^{(4)} = 4x^{-3}(4y/x - 1 - 2y') = q(x, y, y'). \tag{9.3}$$

We shall not need y''' since y' does not occur in (9.1) so we shall not have to use (2.1). We shall be evaluating (9.3) with the variables replaced by intervals. Hence we ought to write the equation in such a way as to obtain sharpest results. The given form is better, for example, than $4x^{-4}\{4y - x(1+2y')\}$.

Equations (8.1), (8.2), and (9.2) dictate that we seek bounds of the form $Y_i = U$ and $Y_i' = V$. We thus replace y by U and y' by V in (9.3) and replace x by suitable intervals. To reduce the labour, we have used intervals X_i^* to find B_i rather than use X_i and X_{i-1} separately. In general practice this should not be done since

$$q(X_{i-1}, U, V) \cup q(X_i, U, V) \subset q(X_i^*, U, V);$$

that is, the left-hand member of this relation yields sharper results, usually. We find

$$B_1 = q(X_1^*, U, V) = [-0{\cdot}500, -0{\cdot}210] + U + V,$$

$$B_2 = q(X_2^*, U, V) = [-0{\cdot}315, -0{\cdot}148] + 0{\cdot}541U + 0{\cdot}630V.$$

Substituting these results into (2.3) and using (9.2), we obtain

$$-900y_1 + 441y_2 = [-21{\cdot}3, -21{\cdot}0] + 0{\cdot}454U + 0{\cdot}454V,$$

$$288y_1 - 585y_2 = [-12{\cdot}1, -12{\cdot}0] + 0{\cdot}161U + 0{\cdot}187V.$$

Solving these equations, we get

$$y_1^I = [0{\cdot}0439, 0{\cdot}0449] + 0{\cdot}000843U + 0{\cdot}000875V,$$

$$y_2^I = [0{\cdot}0419, 0{\cdot}0429] + 0{\cdot}000690U + 0{\cdot}000753V.$$

We next find $\quad Y_0'' = [-0{\cdot}500, -0{\cdot}428] + 0{\cdot}500U$

using (4.2), and

$$Y_0' = [0{\cdot}0483, 0{\cdot}219] + 0{\cdot}170U + 0{\cdot}00263V$$

using (4.3). Similarly, we find Y_1' and Y_2'. Next we obtain

$$Y_0 = [-0{\cdot}0148, 0{\cdot}0680]+0{\cdot}0571U+0{\cdot}00132V$$

as well as Y_1 and Y_2 using (5.2).

Writing (8.6)–(8.9) for $i = 0$, 1, and 2, we have

$$\begin{aligned}
\Delta y_0^L &= -0{\cdot}0148+0{\cdot}9429u-0{\cdot}00132v,\\
\Delta y_1^L &= 0{\cdot}018+0{\cdot}9564u-0{\cdot}00245v,\\
\Delta y_2^L &= -0{\cdot}0113+0{\cdot}9676u-0{\cdot}00113v,\\
\Delta y_0^R &= 0{\cdot}068+0{\cdot}9429u-0{\cdot}00132v,\\
\Delta y_1^R &= -0{\cdot}0688+0{\cdot}9564u-0{\cdot}00245v,\\
\Delta y_2^R &= -0{\cdot}0537+0{\cdot}9676u-0{\cdot}0113v,\\
\Delta z_0^L &= 0{\cdot}0483-0{\cdot}17u+0{\cdot}99737v\\
\Delta z_1^L &= -0{\cdot}0805-0{\cdot}128u+0{\cdot}99511v,\\
\Delta z_2^L &= -0{\cdot}192-0{\cdot}096u+0{\cdot}99774v,\\
\Delta z_0^R &= -0{\cdot}219-0{\cdot}17u+0{\cdot}99737v,\\
\Delta z_1^R &= -0{\cdot}0685-0{\cdot}128u+0{\cdot}99511v,\\
\Delta z_2^R &= 0{\cdot}0635-0{\cdot}096u+0{\cdot}99774v.
\end{aligned} \tag{9.4}$$

Thus (8.18) and (8.19) become (approximately)

$$0 = -51{\cdot}5+390u-v$$

and

$$0 = -0{\cdot}22-0{\cdot}17u+v,$$

respectively. Rounding to one significant digit (for convenient hand calculation) we approximate (8.21) by

$$v = 200u-30. \tag{9.5}$$

It does not matter that we approximate (8.21) so poorly since we choose not to compute u_1 and v_1. Instead we use the alternative method described above.

Substituting for v from (9.5) into (9.4) we find $\Delta y_i^L \geqslant 0$, $\Delta y_i^R \leqslant 0$, $\Delta z_i^L \geqslant 0$, and $\Delta z_i^R \leqslant 0$ for $i = 0$, 1, and 2 if $u = 0{\cdot}152$. If we had used (8.22), we would have found $u_0 = 0{\cdot}133$. The alternative method (which is better) has yielded a worse result because we rounded (8.21) so drastically to get (9.5). Using higher-precision arithmetic to obtain (9.5), we could have got $u_0 = 0{\cdot}133$. Using $u = 0{\cdot}152$, equation (8.1) reveals that $y \in [-0{\cdot}152, 0{\cdot}152]$ for all $x \in [2, 3]$. Solving (9.5) for v and using (8.2), we find that $y' \in [-0{\cdot}4, 0{\cdot}4]$.

We could now use our iterative process beginning with step (*b*) in section 6. However, we already know what the result of performing steps (*b*)–(*e*) will be. Except for the fact that the arithmetic might differ slightly, we would obtain the improvements given by (9.4) for the bounds. We thus find $Y_0 = [-0{\cdot}023, 0{\cdot}0773]$, $Y_1 = [0{\cdot}01, 0{\cdot}765]$, $Y_2 = [-0{\cdot}017, 0{\cdot}0591]$, $Y'_0 = [0{\cdot}021, 0{\cdot}246]$, $Y'_1 = [-0{\cdot}102, 0{\cdot}09]$, and $Y'_2 = [-0{\cdot}208, -0{\cdot}048]$.

We now begin our iterative process. We find $y_1^I = [0{\cdot}0434, 0{\cdot}0447]$ and $y_2^I = [0{\cdot}0418, 0{\cdot}0427]$. As before we have evaluated B_1 and B_2 in the form $B_i = q(X_i^*, Y_{i-1} \cup Y_i, Y'_{i-1} \cup Y'_i)$. To improve sharpness in the final time through the iterative process, we use

$$B_i = q(X_{i-1}, Y_{i-1}, Y'_{i-1}) \cup q(X_i, Y_i, Y'_i).$$

We find $y_1^I = [0{\cdot}0440, 0{\cdot}0446]$ and $y_2^I = [0{\cdot}0422, 0{\cdot}0427]$. Very little improvement could be obtained by further iteration.

Our step-size h is too large to yield high accuracy. However, if we consider the mid-points of y_1^I and y_2^I to be approximate values of y_1 and y_2, we know that the relative errors are less than 0·007 and 0·006, respectively.

Collatz [1] solved this same problem approximately and using an explicit expression for $y^{(4)}$ in terms of x obtained estimates

$$y_1^I = [0{\cdot}043288, 0{\cdot}044708] \quad \text{and} \quad y_2^I = [0{\cdot}041464, 0{\cdot}042884].$$

In practice, of course, we do not know $y^{(4)}$ explicitly in terms of x alone. Without this information, we have obtained error bounds, not estimates, which are sharper.

10. Crude bounds for non-linear equations

We now consider ways in which crude bounds on y and y' can be obtained for non-linear differential equations of the form (1.1) with boundary conditions given by (1.3).

It should be noted that no initial bounds on y' are required if $f(x, y, y')$ in (1.1) is independent of y'. In this case (4.3) provides bounds on y' assuming bounds on y are known.

In very special situations initial bounds may be quite simple to obtain. Suppose, for example, the differential equation is

$$y''(x) = \frac{g(x)}{1+\{y(x)\}^2+\{y'(x)\}^2}.$$

Suppose $g(x)$ is bounded for $x \in [a, b]$. Evaluating $g([a, b])$ in interval arithmetic, let G be the interval obtained. Then $y''(x) \in G$ for all $x \in [a, b]$. Hence (4.3) provides bounds Y'_i $(i = 0, \ldots, n-1)$. We merely let $h = b-a$

so that $y_i = y(a)$ and $y_{i+1} = y(b)$. These quantities are given by (1.3). Similarly, (5.2) yields a bound on $y(x)$ for all $x \in [a,b]$.

In problems for which it is applicable, a crude form of the method discussed by Moore in Chapter 6 could be used to get initial bounds on y.

We now quote a theorem due to Gendzhoian [3] which can be useful:

THEOREM. *Given* $y'' = f(x,y,y')$, $y(0) = y(1) = 0$. *Assume that for* $0 \leqslant x \leqslant 1$ *and* $y^2+y'^2 < \infty$, *the following conditions hold*:

(i) f *is continuous in* x, y, *and* y'.

(ii) f *is continuously differentiable with respect to* y *and* y'.

(iii) $0 \leqslant f_y \leqslant M$ *and* $|f_{y'}| < M$.

Let $N \geqslant 0$ *be such that* $|f(x,0,0)| < 2e^{N/2}$ *and let* $\alpha = \frac{1}{2}\{M+(M^2+4)^{1/2}\}$ *and* $R = \max(N,\alpha)$. *Then* $-v(x) \leqslant y(x) \leqslant v(x)$ *for* $0 \leqslant x \leqslant 1$ *where*

$$v(x) = 1+e^R-e^{Rx}-e^{R(1-x)}.$$

In certain cases, it can be determined that the conditions of this theorem hold. Note that an upper bound for the constant N can be obtained by evaluating $f(x,0,0)$ in interval arithmetic with x replaced by the interval $[0,1]$.

If for all $x \in [a,b]$, the function $f(x,y,y')$ does not grow too rapidly as a function of y and y', we can obtain crude bounds on y and y' in a way similar to that of the last section.

Denote $Y = [y^L, y^R]$, $Y' = [z^L, z^R]$. If we substitute the fixed interval $X = [a,b]$ for x and the variable intervals Y for y and Y' for y' in $f(x,y,y')$, we have

$$f(X,Y,Y') = [g^L, g^R], \tag{10.1}$$

where g^L and g^R are functions of y^L, y^R, z^L, and z^R.

From (4.3), $y' \in \bar{Y}'$ for $x \in X$ where

$$\bar{Y}' = \frac{y(b)-y(a)}{b-a}+\frac{b-a}{2}w\{[0,1][g^L,g^R]\}[-1,1]. \tag{10.2}$$

Thus from (5.2), $y \in \bar{Y}$ for $x \in X$ where

$$\begin{aligned}\bar{Y} &= \tfrac{1}{2}\{y(a)+y(b)\}+\frac{b-a}{2}w([0,1]\bar{Y}')[-1,1] \\ &= \tfrac{1}{2}\{y(a)+y(b)\}+\tfrac{1}{2}\{y(b)-y(a)+(b-a)^2w([0,1][g^L,g^R])\}[-1,1].\end{aligned} \tag{10.3}$$

Denote $\bar{Y} = [\bar{y}^L, \bar{y}^R]$, $\bar{Y}' = [\bar{y}^L, \bar{y}^R]$, $\Delta y^L = \bar{y}^L-y^L$, $\Delta y^R = \bar{y}^R-y^R$, $\Delta z^L = \bar{z}^L-z^L$, and $\Delta z^R = \bar{z}^R-z^R$.

The bounds (10.2) and (10.3) are strictly better than the bounds Y and Y', respectively, if

$$\Delta y^L > 0, \qquad \Delta y^R < 0, \qquad \Delta z^L > 0, \qquad \Delta z^R < 0. \tag{10.4}$$

This will be the case if g^L and g^R grow at less than a linear rate as functions of their arguments provided that $-y^L$, y^R, $-z^L$, and z^R are sufficiently large positive numbers. In particular cases, linear growth of g^L and g^R may be acceptable.

For a linear differential equation, $\bar{y}^L$, $\bar{y}^R$, $\bar{z}^L$, and $\bar{z}^R$ are linear functions of y^L, y^R, z^L, and z^R. Hence it was necessary (in general) in the last section to subdivide X and use (2.3) to assure that, corresponding to (10.4), we could satisfy (8.10). With the assumption that g^L and g^R grow sufficiently slowly, the additional step is unnecessary.

The following example illustrates the ideas just discussed. We develop the results in a way that might occur in practice. That is we impose conditions as they appear necessary or convenient.

Consider the problem

$$y'' = 40(yy')^{1/3}, \qquad y(1) = 40, \qquad y(2) = 320$$

whose solution is $y = 40x^5$. Note that $f(x, y, y') = 40(yy')^{1/3}$ is independent of x. This simplification is neither necessary nor particularly helpful. Substituting Y for y and Y' for y' in f, we obtain $Y'' = 40(YY')^{1/3}$.

From (10.2),

$$\overline{Y}' = 280+20w\{[0,1]Y^{1/3}(Y')^{1/3}\}[-1,1].$$

For convenience, assume $0 \in Y$ and $0 \in Y'$. Then

$$\overline{Y}' = 280+20w\{[(y^L)^{1/3}, (y^R)^{1/3}][(z^L)^{1/3}, (z^R)^{1/3}]\}[-1,1].$$

Since $y(1) > 0$ and $y(2) > 0$, assume $y^R \geqslant -y^L$. Since $y(2)-y(1) > 0$, assume $z^R \geqslant -z^L$. Then

$$[(y^L)^{1/3}, (y^R)^{1/3}][(z^L)^{1/3}, (z^R)^{1/3}]$$
$$=[\min\{(y^Lz^R)^{1/3}, (y^Rz^L)^{1/3}\}, (y^Rz^R)^{1/3}] \subset [-(y^Rz^R)^{1/3}, (y^Rz^R)^{1/3}].$$

Since we may enlarge $\overline{Y}'$ if we like, we accept

$$\overline{Y}' = 280+40(y^Rz^R)^{1/3}[-1,1]. \tag{10.5}$$

From (10.3), we obtain

$$\overline{Y} = [40, 320]+40(y^Rz^R)^{1/3}[-1,1]. \tag{10.6}$$

Hence

$$\Delta y^L = 40-40(y^Rz^R)^{1/3}+y^L,$$
$$\Delta y^R = 320+40(y^Rz^R)^{1/3}-y^R.$$

Choose $y^L = -y^R$. Then $\Delta y^L > 0$ and $\Delta y^R < 0$ if

$$y^R-40(y^Rz^R)^{1/3}-320 > 0. \tag{10.7}$$

From (10.5),

$$\Delta z^L = 280-40(y^Rz^R)^{1/3}-z^L,$$
$$\Delta z^R = 280+40(y^Rz^R)^{1/3}-z^R.$$

Choose $z^L = -z^R$. Then $\Delta z^L > 0$ and $\Delta z^R < 0$ if

$$z^R - 40(y^R z^R)^{1/3} - 280 > 0. \tag{10.8}$$

Choose $z^R = y^R$. Then both (10.7) and (10.8) are satisfied if

$$y^R - 40(y^R)^{2/3} - 320 > 0. \tag{10.9}$$

The largest root of this cubic in $(y^R)^{1/3}$ is near 40·2 and (10.9) is satisfied if $(y^R)^{1/3} \geqslant 40{\cdot}2$. Since $(40{\cdot}2)^3 < 64965$, we conclude that inequalities (10.4) are satisfied if $-y^L = y^R = -z^L = z^R \geqslant 64965$. Using the argument applied in the last section, we conclude that both y and y' are contained in the interval $64965[-1, 1]$ for all $x \in X$. The best possible bounds on y and y' are $[40, 320]$ and $[200, 3200]$, respectively.

Thus the bounds are not very good. However, subdividing X and using these crude bounds in the iterative method described in section 6, good bounds can be obtained.

11. Additional notes

We have shown how, under certain conditions, strict bounds can be obtained on the solution of a two-point boundary-value problem. We not only get bounds on the value of the solution at the mesh points but also uniform bounds on the solution between mesh points. If desired, the method could be easily extended to yield interval polynomial bounds between the mesh points.

We have implicitly assumed that a and b were rational numbers that can be expressed in single precision in the computer. If this is not the case, then h, x_1, x_2,... are irrational, in general. These numbers could be replaced by intervals. However, it seems easier to replace x by, say, $x = a+(b-a)t$. Then t takes the values 0 and 1 at the end-points of the interval in which the differential equation is to be solved. Alternatively, we could choose h to be rational and let only x_1-a and/or $b-x_{n-1}$ be irrational. In this case alternative expressions for (2.1) and (2.2) must be written for $i = 1$ and/or $i = n-1$.

Equations (2.1) and (2.2) are commonly replaced by alternative expressions in practice (see [2]). If the necessary extra derivatives of f can be easily obtained, it seems probable that higher order approximations should be used. Similarly (4.1) and (5.1) could be replaced. For example, suppose the boundary conditions are of the form $y'(a) = y_0'$ and $y'(b) = y_n'$. Then in the crude error bounding method in section 10, it may be better to use, say, an interval version of

$$y'(x) = \tfrac{1}{2}\{y'(a)+y'(b)+(b-x)y''(\theta)-(x-a)y''(\phi)\}$$

in place of (10.2).

To avoid use of $y^{(4)}$, we could replace (2.2) by

$$y_i'' = h^{-2}(y_{i+1}-2y_i-y_{i-1})+\frac{h^3}{6}\{y'''(\xi_i)-y'''(\eta_i)\}, \tag{11.1}$$

where $\xi_i \in X_i$ and $\eta_i \in X_{i-1}$. If this equation is used instead of (2.2), we can drop the condition that y have a bounded fourth derivative.

Equation (11.1) may be especially useful in obtaining crude error bounds. For example, consider the differential equation

$$y'' = y'+\sin y.$$

For this example, we find

$$y''' = y'(1+\cos y)+\sin y$$

and $$y^{(4)} = y'(1+2\cos y)-(y')^2\sin y+(1+\cos y)\sin y.$$

In order to use the procedure in section 10, we could replace

$$Y_i'' = f(X_i, Y_i, Y_i') = Y_i'+\sin Y_i$$

by $\bar{Y}_i'' = Y_i'+[-1,1]$ since $\sin y_i \in [-1,1]$. Similarly, we could replace

$$Y_i''' = p(X_i, Y_i, Y_i') = Y_i'(1+\cos Y_i)+\sin Y_i$$

by $\bar{Y}_i''' = [0,2]Y_i'+[-1,1]$. However, if we attempt to do this for $y^{(4)}$, the result is not linear in Y_i and Y_i'. Hence the method in section 10 could not be used. But if we replace (2.2) by (11.1), we do not require a bound on $y^{(4)}$. The bounds on y_i'' and y_i''' are linear in Y_i and Y_i' and hence we can apply the crude bounding procedure of section 8.

The procedure whose steps are listed in section 6 yields bounds on y' over the intervals X_i. If bounds on y' at the mesh-points x_i are desired, we obtain sharper results by noting that $y_i' \in Y_{i-1}' \cap Y_i'$. Sharper results, in general, can be obtained using (2.1), which becomes

$$y_i' \in \frac{1}{2h}(y_{i+1}^I - y_i^I) - \frac{h^2}{6}A_i.$$

Note we can attempt to improve y_i^I ($i = 1,\ldots, n-1$) by replacing y_i^I by $y_i^I \cap Y_i \cap Y_{i-1}$.

REFERENCES

1. Collatz, L. *The numerical treatment of differential equations*, 3rd edn. Springer Verlag, Berlin (1966).

2. Fox, L. *The numerical solution of two-point boundary problems in ordinary differential equations*. Clarendon Press, Oxford (1957).

3. Gendzhoian, G. V. On the two-sided Chaplygin approximations of the solution of two-point boundary problems (in Russian). *Izv. Akad. Nauk armyan. SSR* **17**, 21–6 (1964).

4. Hansen, Eldon. On solving systems of equations using interval arithmetic. *Math. Comput.* **22**, 374–84 (1968).

5. —— and Smith, Roberta. Interval arithmetic in matrix computations, Part II. *SIAM Jl numer. Anal.* **4**, 1–9 (1967).

6. Moore, Ramon E. *Interval analysis.* Prentice-Hall, New Jersey (1966).

7. Smith, Roberta and Hansen, Eldon. A computer program for solving a system of linear equations and matrix inversion with automatic error bounding using interval arithmetic. *Lockheed Missiles and Space Co. report LMSC* 4–22–66–3 (1966).