`IntLinInc2D` package
User manual

Contents:

1 About the package

Necessary software is MATLAB®.
The package implements the boundary intervals method [1].
Author of `IntLinInc2D` and the boundary intervals method is Irene A. Sharaya (Institute of Computational Technologies SB RAS, Novosibirsk).
The package `IntLinInc2D` is free software. Its source codes are open.
Date of the first release is January 14, 2013.
The latest release is available from `http://interval.ict.nsc.ru/Programing` and `http://interval.ict.nsc.ru/sharaya` .

2 Purpose

The package `IntLinInc2D` is intended to visualize various solution sets for interval and point (i.e., noninterval) systems of relations. These systems and solution sets are listed below.

Interval systems:

1) the set of formal solutions for the interval inclusion

$$\boldsymbol{C}x \subseteq \boldsymbol{d} \tag{1}$$

in Kaucher arithmetic, where
$\boldsymbol{C} = [\underline{C}, \overline{C}] \in \mathbb{KR}^{m\times 2}$ is an interval matrix with given endpoints $\underline{C}$ and $\overline{C}$;
$x \in \mathbb{R}^2$ is a real vector of unknowns;
$\boldsymbol{d} = [\underline{d}, \overline{d}] \in \mathbb{K}\overline{\mathbb{R}}^m$ is an interval vector with given endpoints $\underline{d}$ and $\overline{d}$;
$m \in \mathbb{N}$ is a natural (positive integer) number;
$\mathbb{KR} = \{[\underline{z}, \overline{z}] \mid \underline{z}, \overline{z} \in \mathbb{R}\}$ is the set of Kaucher intervals (in contrast to the set of classical intervals $\mathbb{IR} = \{[\underline{z}, \overline{z}] \mid \underline{z}, \overline{z} \in \mathbb{R},\ \underline{z} \le \overline{z}\}$, the requirement $\underline{z} \le \overline{z}$ is absent for Kaucher intervals);
$\mathbb{K}\overline{\mathbb{R}} = \{[\underline{z}, \overline{z}] \mid \underline{z}, \overline{z} \in \overline{\mathbb{R}}\}$ is the set of Kaucher intervals over the extended real axis $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$;
multiplication $\boldsymbol{C}$ by x is standard for Kaucher arithmetic;
the inclusion "$\subseteq$" is defined by inequalities $\underline{\boldsymbol{C}x} \ge \underline{d}$ and $\overline{\boldsymbol{C}x} \le \overline{d}$, which are understood componentwise, $\underline{\boldsymbol{C}x}$ and $\overline{\boldsymbol{C}x}$ are the left and right endpoints of the interval vector $\boldsymbol{C}x = [\underline{\boldsymbol{C}x}, \overline{\boldsymbol{C}x}]$ respectively;

2) all possible AE-solution sets for the interval system of equations

$$\boldsymbol{A}x = \boldsymbol{b}, \quad \boldsymbol{A} \in \mathbb{IR}^{m\times 2},\ \boldsymbol{b} \in \mathbb{IR}^m,\ m \in \mathbb{N}; \tag{2}$$

3) all possible quantifier solution sets for the interval system of inequalities

$$\boldsymbol{A}x \ge \boldsymbol{b}, \quad \boldsymbol{A} \in \mathbb{IR}^{m\times 2},\ \boldsymbol{b} \in \mathbb{IR}^m,\ m \in \mathbb{N}, \tag{3}$$

or

$$\boldsymbol{A}x \le \boldsymbol{b}, \quad \boldsymbol{A} \in \mathbb{IR}^{m\times 2},\ \boldsymbol{b} \in \mathbb{IR}^m,\ m \in \mathbb{N}; \tag{4}$$

4) various quantifier solution sets for the interval mixed system of linear equations and inequalities

$$\boldsymbol{A}x \ \sigma \ \boldsymbol{b}, \quad \boldsymbol{A} \in \mathbb{IR}^{m\times 2},\ \boldsymbol{b} \in \mathbb{IR}^{m},\ \sigma \in \{=, \geq, \leq\}^m,\ m \in \mathbb{N}; \tag{5}$$

specifically, we mean all those solutions for which quantifier description has AE-order of quantifiers for rows with the relation "=".

Point systems:

1) the solution set for the system

$$Ax + B|x| \geq c, \quad A, B \in \mathbb{R}^{m\times 2},\ c \in \mathbb{R}^m,\ m \in \mathbb{N}; \tag{6}$$

2) the solution set for the system

$$|Ax - c| \leq B|x| + d, \quad A, B \in \mathbb{R}^{m\times 2},\ c, d \in \mathbb{R}^m,\ m \in \mathbb{N}; \tag{7}$$

3) the solution set for the mixed system of linear equations, inequalities and two-sided inequalities

$$\begin{cases} A_{(1)}x = b_{(1)}, & A_{(1)} \in \mathbb{R}^{m_1\times 2},\ b_{(1)} \in \mathbb{R}^{m_1}, & m_1 \in \mathbb{N}\cup\{0\}, \\ b_{(2)} \leq A_{(2)}x, & A_{(2)} \in \mathbb{R}^{m_2\times 2},\ b_{(2)} \in \mathbb{R}^{m_2}, & m_2 \in \mathbb{N}\cup\{0\}, \\ A_{(3)}x \leq b_{(3)}, & A_{(3)} \in \mathbb{R}^{m_3\times 2},\ b_{(3)} \in \mathbb{R}^{m_3}, & m_2 \in \mathbb{N}\cup\{0\}, \\ b_{(4)} \leq A_{(4)}x \leq b_{(5)}, & A_{(4)} \in \mathbb{R}^{m_4\times 2},\ b_{(4)}, b_{(5)} \in \mathbb{R}^{m_4}, & m_4 \in \mathbb{N}\cup\{0\}, \end{cases} \tag{8}$$

with $m_1 + m_2 + m_3 + m_4 > 0$.

In [2], it is shown that each solution set listed above can be represented as the set of formal solutions to the inclusion (1). Therefore, the visualization of this set play a key role in the package, which is reflected in the title `IntLinInc2D`, i. e. Interval Linear Inclusion. The last letters `2D` mean that the dimension of the unknowns is 2 ($x \in \mathbb{R}^2$).

Remark. The package `IntLinInc2D` is aimed at illustrating simple examples (in publications, education, etc.), so it works most correctly when the initial data are integers and lie in the range $[-10^2, 10^2]$.

3 Structure

The main function of the package is `Cxind2D`. It is designed to visualize the set of formal solutions for the inclusion (1).

The functions used in the main one are

```
BoundaryIntervals,  Intervals2Path,
ClearRows,          NonRepeatRows,
CutBox,             OrientationPoints,
DrawingBox,         SSinW.
```

The package contains auxiliary functions for the problems equivalent to (1). The choice of the auxiliary function depends on which of the systems (2)–(7) is to be processed and, for interval systems, on the solution type. The names of the auxiliary functions reflect this dependency.

The names of the auxiliary functions for the interval systems

system	solution type				
	weak	tolerable	controllable	strong	quantifier
(2) $\boldsymbol{A}x = \boldsymbol{b}$	`EqnWeak2D`	`EqnTol2D`	`EqnCtl2D`	`EqnStrong2D`	`EqnAEss2D`
(3) $\boldsymbol{A}x \geq \boldsymbol{b}$	`GeqWeak2D`	`GeqTol2D`	`GeqCtl2D`	`GeqStrong2D`	`GeqQtr2D`
(4) $\boldsymbol{A}x \leq \boldsymbol{b}$	`LeqWeak2D`	`LeqTol2D`	`LeqCtl2D`	`LeqStrong2D`	`LeqQtr2D`
(5) $\boldsymbol{A}x\ \sigma\ \boldsymbol{b}$	`MixWeak2D`	`MixTol2D`	`MixCtl2D`	`MixStrong2D`	`MixQtr2D`

The solution types from the table above, except the quantifier type, will be defined in this manual. A complete set of definitions is in [2], and it generalizes the terminology from [3, 4]. Note that the auxiliary functions `EqnAEss2D` and `MixQtr2D` are designed only for such quantifier solutions which have AE-order of quantifiers in rows with the relation "=".

For point systems, there are two auxiliary functions: the function `Abs12D` is intended for the system (6) with one absolute value operation, the function `Abs22D` is designed for the system (7) which contains two such operations.

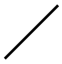
Arguments of the main and auxiliary functions are described in comments within their bodies. To see these descriptions in MATLAB command window, use command `help`, for example,

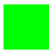
```
>> help EqnWeak2D
```

4 Notation in figures

By po_k (piece in orthant k), we denote the intersection of the solution set with the k-th orthant, $k = 1, 2, 3, 4$. Then

 is a vertex of po_k (orientation point),

 is an edge of po_k,

 is interior of po_k for certain k.

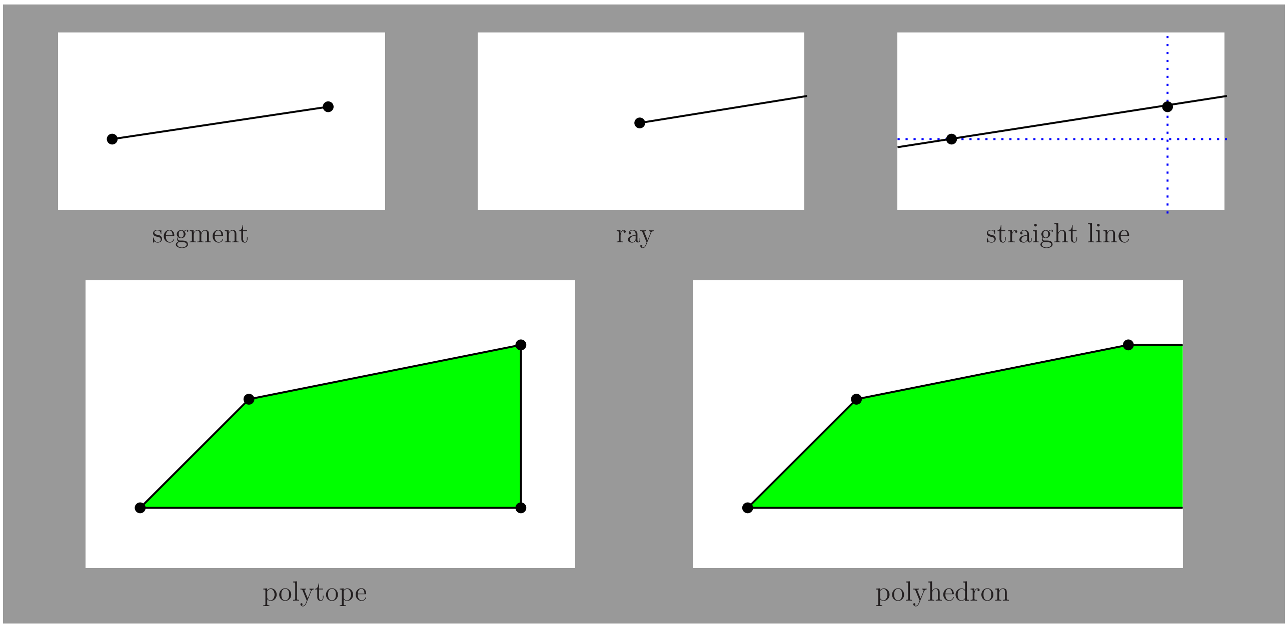
Dotted lines are coordinate axes passing from the origin of coordinates. Abscissa axis corresponds to the variable x_1, ordinate axis corresponds to the variable x_2.

The package chooses the drawing area in such a way that

1) all the orientation points are visible,

2) bounded solution set lies in the interior of the drawing area, in contrast to unbounded solution set that has points on the border of the drawing area.

For instance:



5 Examples

5.1 The set of formal solutions for the interval inclusion $\boldsymbol{C}x \subseteq \boldsymbol{d}$

A vector $x \in \mathbb{R}^2$ is said to be a *formal solution for* (1) if multiplying $\boldsymbol{C}$ by x in Kaucher interval arithmetic produces such an interval vector $\boldsymbol{C}x$ that the inequalities $\underline{\boldsymbol{C}x} \geq \underline{d}$ and $\overline{\boldsymbol{C}x} \leq \overline{d}$ hold.

Example. To see the set of formal solutions for the inclusion

$$\begin{pmatrix} 1 & 0 \\ [1,-1] & [1,3] \end{pmatrix} x \subseteq \begin{pmatrix} [-3,3] \\ [2,3] \end{pmatrix}.$$

How to use the package? In this example, we have for (1):

$$\underline{C} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ \overline{C} = \begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}, \ \underline{d} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \ \overline{d} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

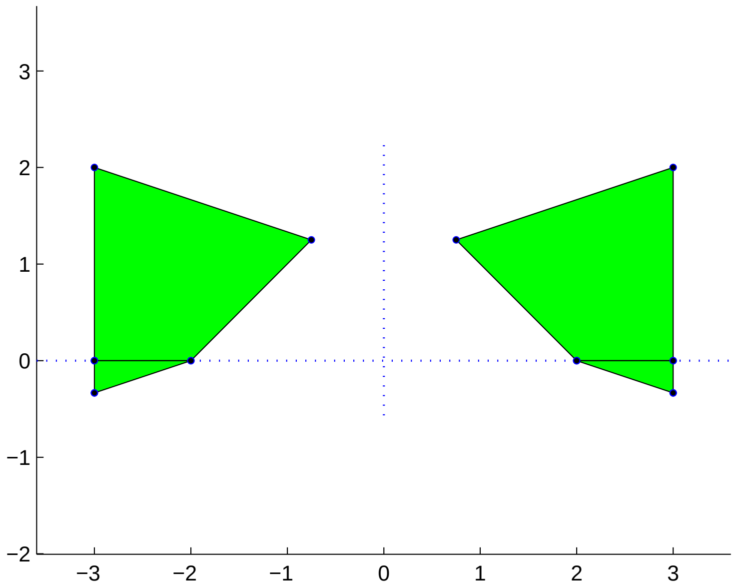
We input this data sequentially and call the main function `Cxind2D`:

```
>> uC=[ 1 0 ; 1 1 ];
>> oC=[ 1 0 ; -1 3 ];
>> ud=[ -3 ; 2 ];
>> od=[  3 ; 3 ];
>> [V,P1,P2,P3,P4]=Cxind2D(uC,oC,ud,od)
```

The output is

```
Number of orientation points = 10

V =

    2.00     3.00     0.75     3.00    -3.00    -0.75    -3.00    -2.00    -3.0000     3.0000
    0        2.00     1.25     0        0         1.25     2.00     0       -0.3333    -0.3333

P1 =

    2.0000         0
    0.7500    1.2500
    3.0000    2.0000
    3.0000         0
    2.0000         0
```

```
P2 =

   -3.0000         0
   -3.0000    2.0000
   -0.7500    1.2500
   -2.0000         0
   -3.0000         0

P3 =

   -3.0000   -0.3333
   -3.0000         0
   -2.0000         0
   -3.0000   -0.3333

P4 =

    3.0000   -0.3333
    2.0000         0
    3.0000         0
    3.0000   -0.3333
```



In the above, V is the matrix of orientation points. It consists of vertices of intersections of the solution set with orthants. Pk is a closed clockwise path around the piece po_k of the solution set in the k-th orthant.

Note. If we change the command

```
>> [V,P1,P2,P3,P4]=Cxind2D(uC,oC,ud,od)
```

by

```
>> [V]=Cxind2D(uC,oC,ud,od)
```

then Pk will not be displayed. If we use

```
>> Cxind2D(uC,oC,ud,od);
```

then we will receive figure and number of orientation points only. (In MATLAB, semicolon after a command suppresses displaying output arguments.)

Application of output arguments and semicolons in all the auxiliary functions is similar to the above. For brevity, we use short calls of the auxiliary functions (with semicolon) in the rest of examples, so that the outputs contain only figures and messages about numbers of orientation points.

5.2 AE-solution sets for the interval system of equations $\boldsymbol{A}x = \boldsymbol{b}$

We turn to visualization of AE-solution sets for the interval system of linear equations (2).

5.2.1 United solution set (= set of weak solutions)

A vector x is called a *weak solution for* (2), if

$$(\exists A \in \boldsymbol{A})\ (\exists b \in \boldsymbol{b})\ (Ax = b).$$

In other words, a weak solution is a solution to a point system $Ax = b$ for some $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. The weak solution set consists of solution sets to all point systems, therefore it is also referred to as *united solution set*.

Example. To see the united solution set for the system

$$\begin{pmatrix} [-1,1] & [-1,1] \\ -1 & [-1,1] \end{pmatrix} x = \begin{pmatrix} 1 \\ [-2,2] \end{pmatrix}.$$

How to use the package? In this case, the specific data for (2) are

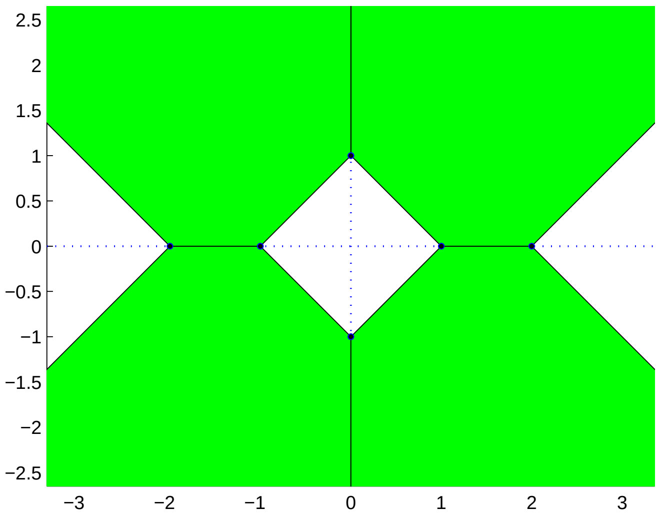
$$\underline{A} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}, \ \overline{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \ \underline{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \ \overline{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We input them sequentially and call the auxiliary function `EqnWeak2D`:

```
>> infA=[ -1 -1 ; -1 -1 ];
>> supA=[ 1 1 ; -1 1 ];
>> infb=[ 1 ; -2 ];
>> supb=[ 1 ; 2 ];
>> EqnWeak2D(infA,supA,infb,supb);
```

On output, we have

```
Number of orientation points = 6
```



5.2.2 Tolerable solution set

A vector x is called a *tolerable solution for the system* (2), if

$$(\forall A \in \boldsymbol{A})\ (\exists b \in \boldsymbol{b})\ (Ax = b).$$

Example. To see the tolerable solution set for the system

$$\begin{pmatrix}[-1,1] & [-1,1]\end{pmatrix} x = \begin{pmatrix}[-1,1]\end{pmatrix}.$$

How to use the package? This example corresponds to the following specific data for the interval system (2):

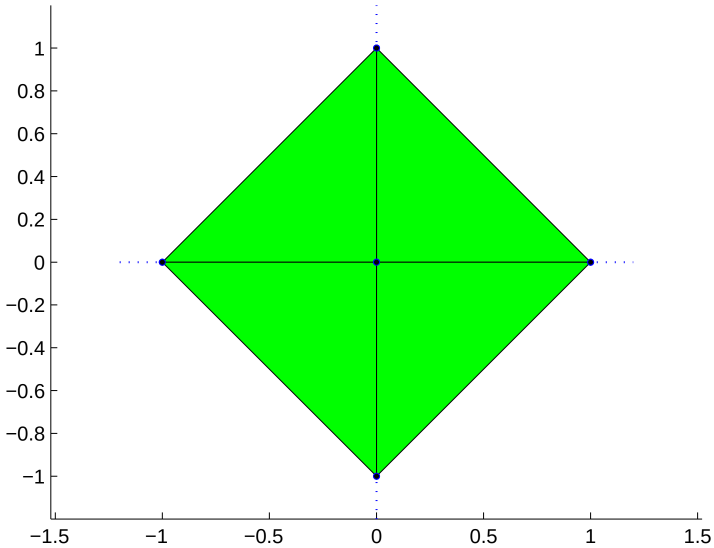
$$\underline{A} = \begin{pmatrix}-1 & -1\end{pmatrix},\ \overline{A} = \begin{pmatrix}1 & 1\end{pmatrix},\ \underline{b} = \begin{pmatrix}-1\end{pmatrix},\ \overline{b} = \begin{pmatrix}1\end{pmatrix}.$$

After inputting them sequentially and calling the auxiliary function `EqnTol2D`

```
>> infA=[ -1 -1 ];
>> supA=[ 1 1  ];
>> infb=[ -1 ];
>> supb=[ 1 ];
>> EqnTol2D(infA,supA,infb,supb);
```

we get

```
Number of orientation points = 5
```



5.2.3 Controllable solution set

A vector x is called a *controllable solution for the system* (2), if

$$(\forall b \in \boldsymbol{b})\ (\exists A \in \boldsymbol{A})\ (Ax = b).$$

Example. To see the controllable solution set for the system

$$\begin{pmatrix}[-1,1] & [-1,1]\end{pmatrix} x = \begin{pmatrix}[-1,1]\end{pmatrix}.$$

How to use the package? The specific data for the interval equation (2) are

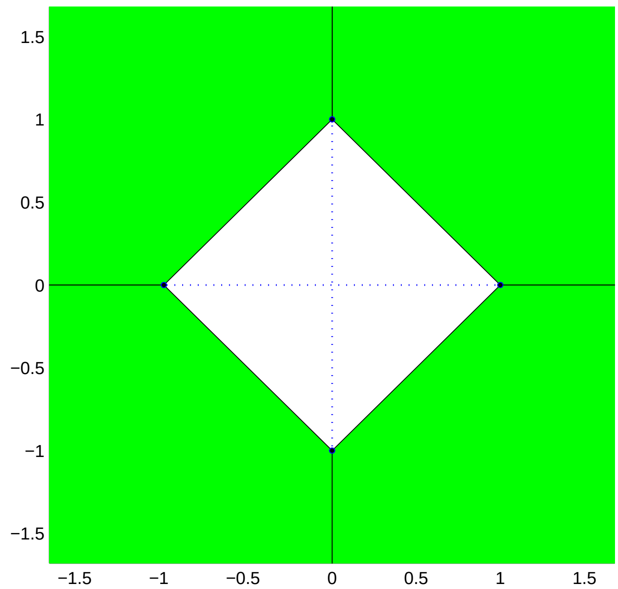
$$\underline{A} = \begin{pmatrix}-1 & -1\end{pmatrix},\ \overline{A} = \begin{pmatrix}1 & 1\end{pmatrix},\ \underline{b} = \begin{pmatrix}-1\end{pmatrix},\ \overline{b} = \begin{pmatrix}1\end{pmatrix}.$$

We input them sequentially and call the auxiliary function `EqnCtl2D`:

```
>> infA=[ -1 -1 ];
>> supA=[ 1 1 ];
>> infb=[ -1 ];
>> supb=[ 1 ];
>> EqnCtl2D(infA,supA,infb,supb);
```

The output is

```
Number of orientation points = 4
```



5.2.4 Strong solution set

A vector x is called a *strong solution for the system* (2), if

$$(\forall A \in \boldsymbol{A})\ (\forall b \in \boldsymbol{b})\ (Ax = b).$$

It is clear from the definition, that the strong solution set for (2) is empty in most cases. But sometimes it is not empty, for example:

- if $\boldsymbol{A} = 0$ and $\boldsymbol{b} = 0$, then the strong solution set is the entire plane;
- if $\boldsymbol{A}$ has exactly one zero column and $\boldsymbol{b} = 0$, then the strong solution set coincides with a coordinate axis;
- if $\boldsymbol{A} = A$ is a point matrix and $\boldsymbol{b} = b$ is a point vector and the point system $Ax = b$ is solvable, then the strong solution set for the interval system $\boldsymbol{A}x = \boldsymbol{b}$ coincides with the solution set for the point system $Ax = b$.

Example. To see the strong solution set for the system

$$\begin{pmatrix} 1 & [3,4] \\ 2 & 5 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

How to use the package? Here the specific data for the system (2) are:

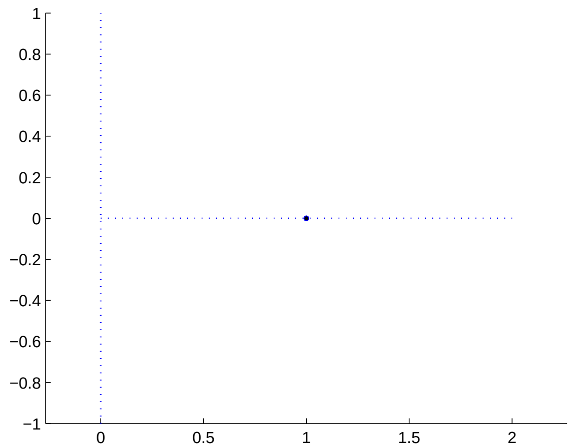
$$\underline{A} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix},\ \overline{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix},\ \underline{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \overline{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Inputting them sequentially and calling the auxiliary function `EqnStrong2D`

```
>> infA=[ 1 3 ; 2 5 ];
>> supA=[ 1 4 ; 2 5 ];
>> infb=[ 1 ; 2 ];
>> supb=[ 1 ; 2 ];
>> EqnStrong2D(infA,supA,infb,supb);
```

produce

```
Number of orientation points = 1
```



5.2.5 Arbitrary AE-solution set

AE-solutions for the interval system of equations (2) were defined in [3].

Example. Let the system (2) take the form

$$\begin{pmatrix} [-1,1] & [-1,1] \\ [-2,2] & [-2,2] \end{pmatrix} x = \begin{pmatrix} [-1,1] \\ [-2,2] \end{pmatrix}.$$

We want to see the following AE-solution set

$$\{x \in \mathbb{R}^2 \mid (\forall A_{11} \in \boldsymbol{A}_{11})(\forall A_{12} \in \boldsymbol{A}_{12})(\forall b_2 \in \boldsymbol{b}_2)$$
$$(\exists A_{21} \in \boldsymbol{A}_{21})(\exists A_{22} \in \boldsymbol{A}_{22})(\exists b_1 \in \boldsymbol{b}_1)\ (Ax = b)\}.$$

How to use the package? This example specifies the data for (2) as follows

$$\underline{A} = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix}, \ \overline{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \ \underline{b} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \ \overline{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

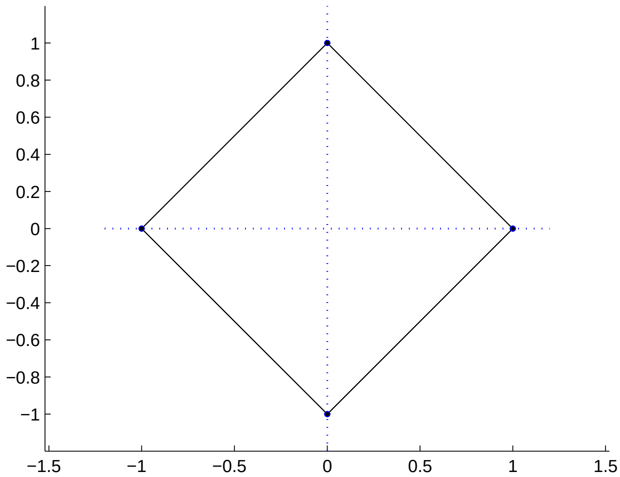
$$A^q = \begin{pmatrix} \forall & \forall \\ \exists & \exists \end{pmatrix}, \quad b^q = \begin{pmatrix} \exists \\ \forall \end{pmatrix}.$$

Let us input this information and call the auxiliary function `EqnAEss2D`:

```
>> infA=[ -1 -1 ; -2 -2 ];
>> supA=[  1  1 ;  2  2 ];
>> infb=[ -1 ; -2 ];
>> supb=[  1 ;  2 ];
>> Aq=[ 'A' 'A' ; 'E' 'E' ];
>> bq=[ 'E' ; 'A' ];
>> EqnAEss2D(infA,supA,Aq,infb,supb,bq);
```

We get:

```
Number of orientation points = 4
```



5.3 Quantifier solution sets for interval systems of linear inequalities ($\boldsymbol{A}x \geq \boldsymbol{b}$ or $\boldsymbol{A}x \leq \boldsymbol{b}$)

We will consider the inequality $\boldsymbol{A}x \geq \boldsymbol{b}$ in Sections 5.3.1 and 5.3.2 and touch on the opposite inequality $\boldsymbol{A}x \leq \boldsymbol{b}$ in Section 5.3.3. For the interval systems of linear inequalities, quantifier solutions and their main types are defined in [2].

5.3.1 Main types of quantifier solutions for the system $\boldsymbol{A}x \geq \boldsymbol{b}$

For the interval system of linear inequalities (3), we call a vector x

a *weak solution,*	if	$(\exists A \in \boldsymbol{A})\ (\exists b \in \boldsymbol{b})\ (Ax \geq b)$,
a *tolerable solution,*	if	$(\forall A \in \boldsymbol{A})\ (\exists b \in \boldsymbol{b})\ (Ax \geq b)$,
a *controllable solution,*	if	$(\forall b \in \boldsymbol{b})\ (\exists A \in \boldsymbol{A})\ (Ax \geq b)$,
a *strong solution,*	if	$(\forall A \in \boldsymbol{A})\ (\forall b \in \boldsymbol{b})\ (Ax \geq b)$.

We regard these solution types as *main types of quantifier solutions* for the interval system of linear inequalities $\boldsymbol{A}x \geq \boldsymbol{b}$.

Example. To see the main types of the quantifier solution sets for the interval inequality

$$\begin{pmatrix}[1,2] & [1,2]\end{pmatrix} x \ \geq\ \begin{pmatrix}[-1,1]\end{pmatrix}.$$

How to use the package? In this case, the specific data for (3) are

$$\underline{A} = \begin{pmatrix}1 & 1\end{pmatrix},\ \overline{A} = \begin{pmatrix}2 & 2\end{pmatrix},\ \underline{b} = \begin{pmatrix}-1\end{pmatrix},\ \overline{b} = \begin{pmatrix}1\end{pmatrix}.$$

We input them sequentially:

```
>> infA=[ 1 1 ];
>> supA=[ 2 2 ];
>> infb=[ -1 ];
>> supb=[  1 ];
```

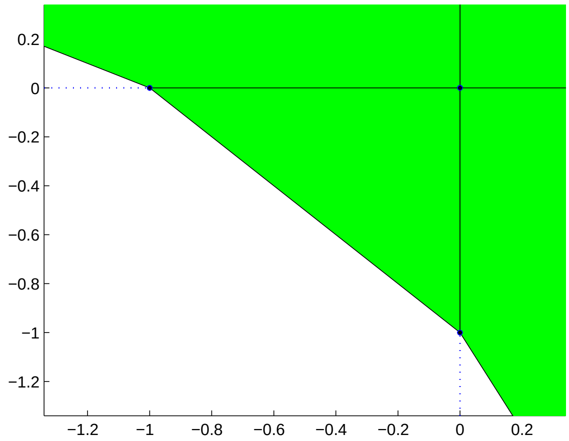
Then, for every main type of the quantifier solution, we call an auxiliary function according to the following table

solution type	**function**
weak	`GeqWeak2D`
tolerable	`GeqTol2D`
controllable	`GeqCtl2D`
strong	`GeqStrong2D`

The output is

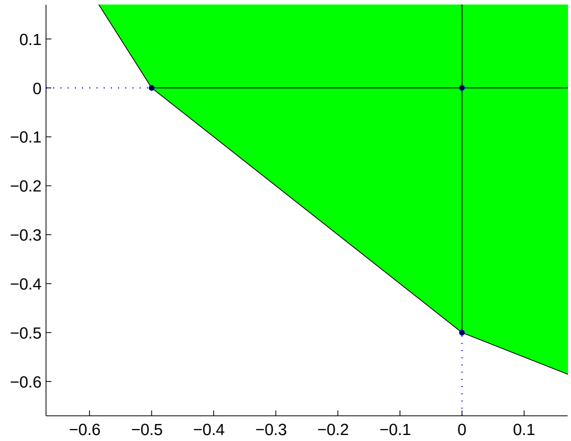
weak solution set

```
>> GeqWeak2D(infA,supA,infb,supb);
Number of orientation points = 3
```



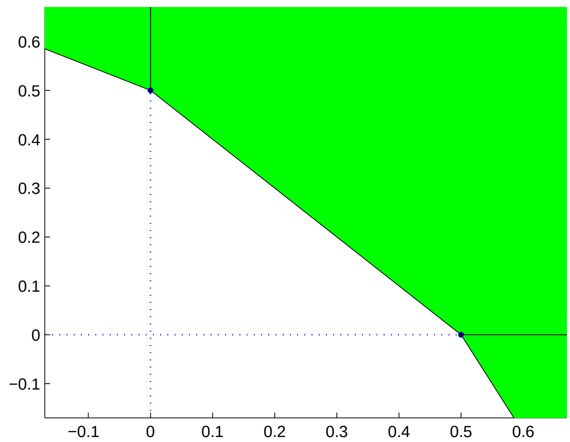
tolerable solution set

```
>> GeqTol2D(infA,supA,infb,supb);
Number of orientation points = 3
```



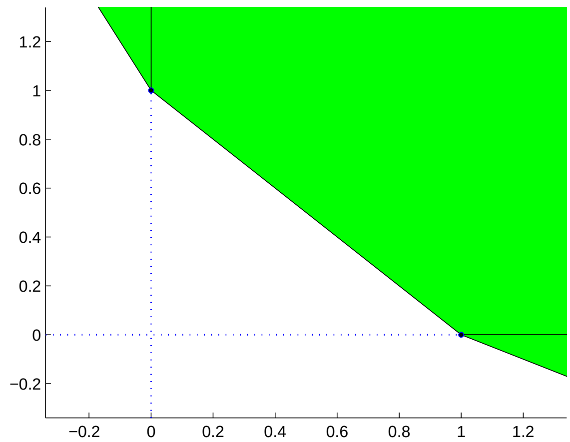
controllable solution set

```
>> GeqCtl2D(infA,supA,infb,supb);
Number of orientation points = 2
```



strong solution set

```
>> GeqStrong2D(infA,supA,infb,supb);
Number of orientation points = 2
```



5.3.2 Arbitrary quantifier solution set for the system $\boldsymbol{A}x \geq \boldsymbol{b}$

Now let us turn to visualization of quantifier solution sets for the system of linear inequalities (3) with arbitrary collection and order of quantifiers.

Example. To see the set

$$\{x \in \mathbb{R}^2 \mid (\exists a_1 \in [1,2])\ (\forall a_2 \in [1,2])\ (\forall b \in [-1,1])\ (a_1x_1 + a_2x_2 \geq b)\}.$$

How to use the package? Reformulating the problem, we need to see the solution set of the interval-quantifier inequality

$$\big([1,2]^{\exists} \quad [1,2]^{\forall}\big)\, x \ \geq \ \big([-1,1]^{\forall}\big)\,.$$

In this example, we have for (3):

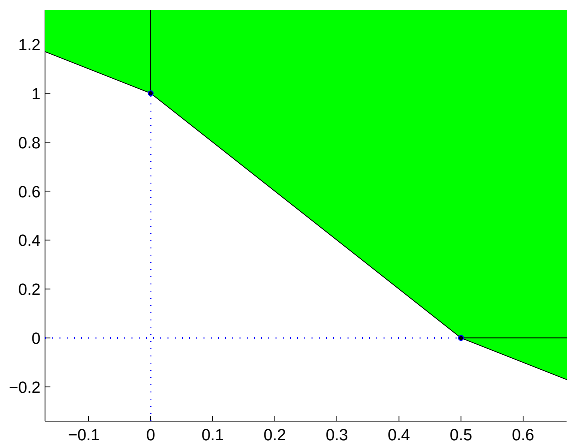
$$\underline{A} = \big(1 \quad 1\big)\,,\ \overline{A} = \big(2 \quad 2\big)\,,\ A^q = \big(\exists \quad \forall\big)\,,\ \underline{b} = \big(-1\big)\,,\ \overline{b} = \big(1\big)\,,\ b^q = \big(\forall\big)\,.$$

Inputting the data and calling the auxiliary function `GeqQtr2D`

```
>> infA=[ 1 1 ];
>> supA=[ 2 2 ];
>> Aq=['E' 'A'];
>> infb=[ -1 ];
>> supb=[  1 ];
>> bq=['A'];
>> GeqQtr2D(infA,supA,Aq,infb,supb,bq);
```

we get

```
Number of orientation points = 2
```



Note. The main types of the quantifier solution sets can be processed in the same way.

5.3.3 Quantifier solution sets for the system $\boldsymbol{A}x \leq \boldsymbol{b}$

The main types of the quantifier solutions for the interval inequality $\boldsymbol{A}x \leq \boldsymbol{b}$ are defined analogously to the opposite inequality $\boldsymbol{A}x \geq \boldsymbol{b}$. The definitions differ only in the inequality sign of the point system: $Ax \leq b$ instead of $Ax \geq b$.

If we want to use the package `IntLinInc2D` for the inequality $\boldsymbol{A}x \leq \boldsymbol{b}$, we must follow the instructions for the opposite inequality $\boldsymbol{A}x \geq \boldsymbol{b}$, but change the prefix `Geq` (greater or equal) for `Leq` (little or equal) in the names of auxiliary functions.

If neccesary, one can look at the examples from Sections 5.3.1 and 5.3.2 for the opposite inequalities. These examples have balanced right-hand side $\boldsymbol{b}$ ($\boldsymbol{b} = -\boldsymbol{b}$). Therefore, if we change the inequality sign, the solution set rotates around the origin of coordinates by 180°.

5.4 Quantifier solution sets for the interval system of relations $\boldsymbol{A}x\,\sigma\,\boldsymbol{b}$

The interval system of relations (5) consists of the interval equations and inequalities of the form $\boldsymbol{A}_{i:}x = \boldsymbol{b}_i$, $\boldsymbol{A}_{i:}x \geq \boldsymbol{b}_i$ and $\boldsymbol{A}_{i:}x \leq \boldsymbol{b}_i$. The presence of all three types of the relations is not obligatory. The interval system of equations (2) and interval systems (3) and (4) of inequalities are particular cases of the relations system (5). The quantifier solution and main types of quantifier solutions for the system $\boldsymbol{A}x\ \sigma\ \boldsymbol{b}$ are defined in [2].

5.4.1 Main types of quantifier solutions

We introduce *main types of quantifier solutions* for the interval mixed system of linear relations (5) by analogy with the interval system of linear equations (or inequalities). A vector $x \in \mathbb{R}^n$ will be referred to as

$$\begin{aligned}
\text{a } \textit{weak solution}, \text{ if } &\ (\exists A \in \boldsymbol{A})\ (\exists b \in \boldsymbol{b})\ (Ax\ \sigma\ b),\\
\text{a } \textit{tolerable solution}, \text{ if } &\ (\forall A \in \boldsymbol{A})\ (\exists b \in \boldsymbol{b})\ (Ax\ \sigma\ b),\\
\text{a } \textit{controllable solution}, \text{ if } &\ (\forall b \in \boldsymbol{b})\ (\exists A \in \boldsymbol{A})\ (Ax\ \sigma\ b),\\
\text{a } \textit{strong solution}, \text{ if } &\ (\forall A \in \boldsymbol{A})\ (\forall b \in \boldsymbol{b})\ (Ax\ \sigma\ b).
\end{aligned}$$

Example. To see the main types of the quantifier solution sets for the interval system of relations

$$\begin{pmatrix} [-1,1] & [-1,1] \\ [-1,1] & 0 \\ 0 & [-1,1] \end{pmatrix} x \begin{pmatrix} = \\ \leq \\ \geq \end{pmatrix} \begin{pmatrix} [-3,3] \\ [-2,2] \\ [-2,2] \end{pmatrix}. \tag{9}$$

How to use the package? The example has the following specific data for (5):

$$\underline{A} = \begin{pmatrix} -1 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix},\ \overline{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\ \underline{b} = \begin{pmatrix} -3 \\ -2 \\ -2 \end{pmatrix},\ \overline{b} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix},\ \sigma = \begin{pmatrix} = \\ \leq \\ \geq \end{pmatrix}.$$

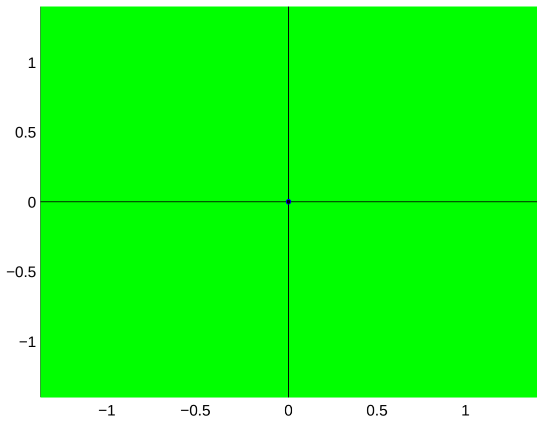
We input them typing

```
>> infA=[-1 -1; -1 0; 0 -1];
>> supA=[ 1  1;  1 0; 0  1];
>> infb=[-3; -2; -2];
>> supb=[ 3;  2;  2];
>> relations=['='; '<'; '>'];
```

Running the corresponding auxiliary functions of the package, we obtain:

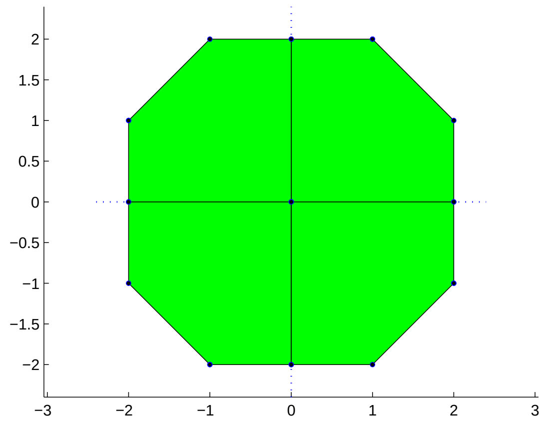
weak solution set

```
>> MixWeak2D(infA,supA,infb,supb,relations);
Number of orientation points = 1
```



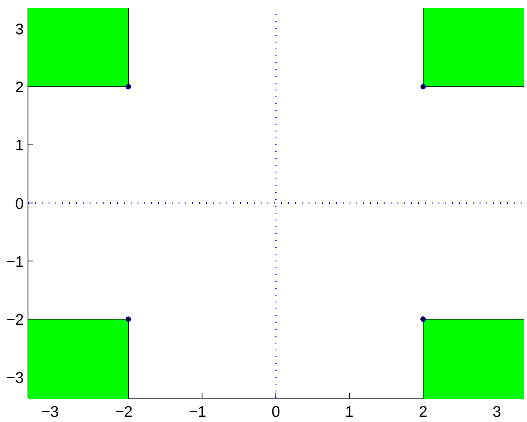
toleralbe solution set

```
>> MixTol2D(infA,supA,infb,supb,relations);
Number of orientation points = 13
```



controllable solution set

```
>> MixCtl2D(infA,supA,infb,supb,relations);
Number of orientation points = 4
```



strong solution set

```
>> MixStrong2D(infA,supA,infb,supb,relations);
Solution set is empty
(it does not have boundary intervals)
```

5.4.2 Quantifier solution set

We turn to the quantifier solution sets for the interval system of relations (5) in the case when the set of quantifiers is arbitrary while their order satisfies the following condition: for the relations with the equality sign, all the prefixes with the universal quantifier "∀" (if such prefixes present in the description of the solution) precede those with the existential quantifiers "∃" (if any).

Example. To see the set of all $x \in \mathbb{R}^2$ satisfying the condition

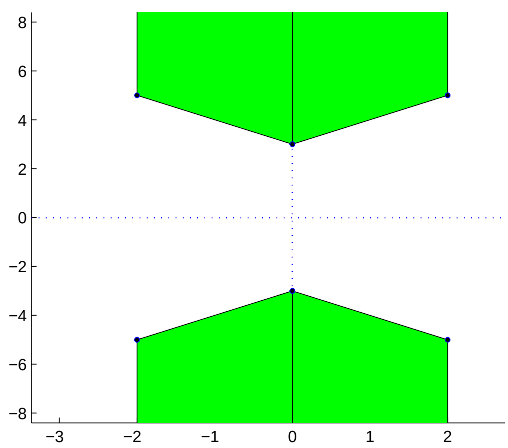
$$\Big((\forall\, b_1 \in [-3,3]) \ (\forall A_{11} \in [-1,1]) \ (\exists A_{12} \in [-1,1]) \ \ (A_{11}x_1 + A_{12}x_2 = b_1) \Big)$$
$$\& \ \Big((\exists\, b_2 \in [-2,2]) \ (\forall A_{21} \in [-1,1]) \ \ (A_{21}x_1 \le b_2) \Big)$$
$$\& \ \Big((\exists\, b_3 \in [-2,2]) \ (\exists A_{32} \in [-1,1]) \ \ (A_{32}x_2 \ge b_3) \Big).$$

How to use the package? The problem is reduced to visualization of quantifier solution set for the interval linear system (9). The above requirement on the order of quantifiers is fulfilled. The system has point elements ($\boldsymbol{A}_{22}$ and $\boldsymbol{A}_{31}$) so it allows freedom in the choice of quantifiers for these elements. Let us fix quantifiers choice selecting matrix and vector of quantifiers respectively as $A^q = \begin{pmatrix} \forall & \exists \\ \forall & \exists \\ \forall & \exists \end{pmatrix}$, $b^q = \begin{pmatrix} \forall \\ \exists \\ \exists \end{pmatrix}$. Inputting the data and calling the function `MixQtr2D`

```
>> infA=[-1 -1; -1 0; 0 -1];
>> supA=[ 1  1;  1 0; 0  1];
>> infb=[-3; -2; -2];
>> supb=[ 3;  2;  2];
>> relations=['='; '<'; '>'];
>> Aq=[ 'A' 'E'; 'A' 'E'; 'A' 'E']
>> bq=['A';'E';'E']
>> MixQtr2D(infA,supA,Aq,infb,supb,bq,relations);
```

produce

```
Number of orientation points = 6
```



5.5 Solution sets for point systems of relations

5.5.1 Solution set for the system $Ax + B|x| \geq c$

Example. To see the solution set for the system (6), where

$$A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

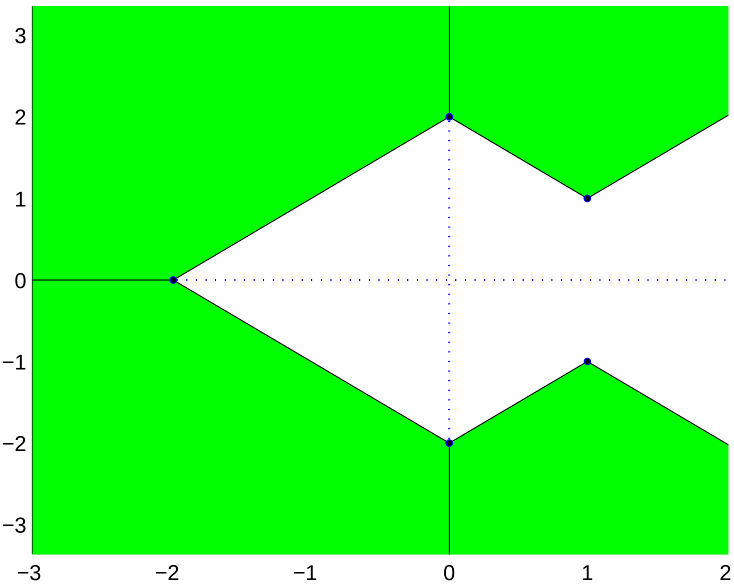
How to use the package?

We input the data sequentially and run the auxiliary function `Abs12D`

```
>> A=[ 0 0 ; -1 0 ];
>> B=[ .5 .5 ; 0 1 ];
>> c=[ 1 ; 0 ];
>> V=Abs12D(A,B,c)
```

The output is

```
Number of orientation points = 5

V =
     1     0    -2     0     1
     1     2     0    -2    -1
```



5.5.2 Solution set for the system $|Ax - c| \leq B|x| + d$

Example. To see the solution set for the system (7) whith

$$A = \begin{pmatrix} 0 & 0 \\ -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

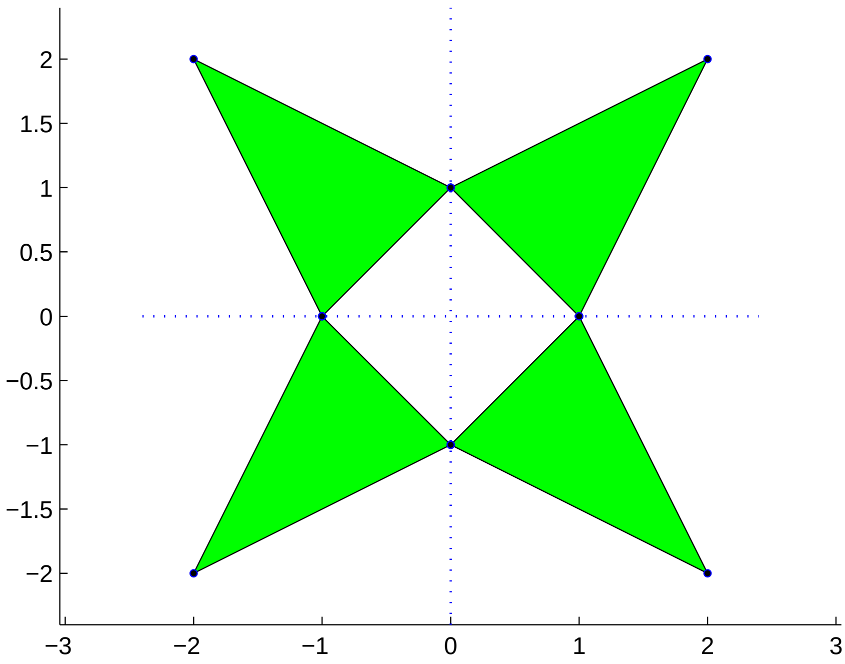
How to use the package?

We input the data and call the function `Abs22D`:

```
>> A=[ 0 0 ; -2 0 ; 0 -2 ];
>> B=[ 1 1 ; 0 1 ; 1 0 ];
>> c=[ 1 ; 0 ; 0 ];
>> d=[ 0 ; 2 ; 2 ];
>> Abs22D(A,B,c,d);
```

The output is

```
Number of orientation points = 8
```



5.6 Solution set for a mixed system of equations and inequalities

The point system (8) is a particular case of the interval inclusion (1) with the following correspondence between their rows:

if the i-th row of the point system is	then the corresponding row $\boldsymbol{C}_{i:}x \subseteq \boldsymbol{d}_i$ has $\underline{C}_{i:} = \overline{C}_{i:} = A_{i:}$ and
$A_{i:}x = b_i$	$\underline{d}_i = \overline{d}_i = b_i$
$b_i \le A_{i:}x$	$\underline{d}_i = b_i,\ \overline{d}_i = \infty$
$A_{i:}x \le b_i$	$\underline{d}_i = -\infty,\ \overline{d}_i = b_i$
$u_i \le A_{i:}x \le v_i$	$\underline{d}_i = -u_i,\ \overline{d}_i = v_i$

Example. To see the solution set for the system

$$\begin{cases} x_1 - x_2 = 0, \\ \quad x_2 \le 1. \end{cases}$$

How to use the package? We have following concrete data for (1):

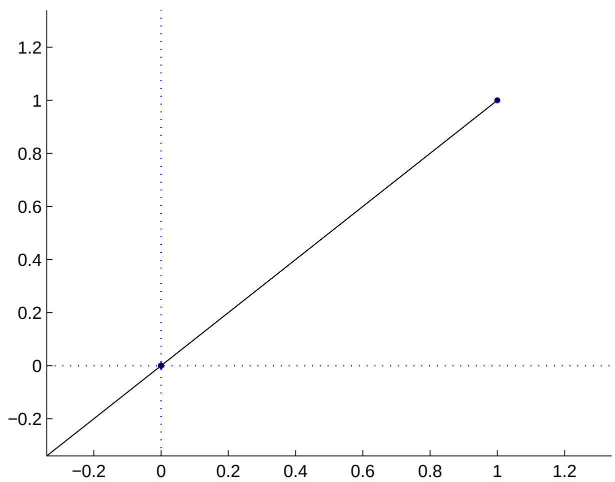
$$\underline{C} = \overline{C} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \ \underline{d} = \begin{pmatrix} 0 \\ -\infty \end{pmatrix}, \ \overline{d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

After inputting these data and calling the main function `Cxind2D`,

```
>> uC=[ 1 -1 ; 0 1 ];
>> oC=uC;
>> ud=[ 0 ; -Inf ];
>> od=[ 0 ;  1 ];
>> Cxind2D(uC,oC,ud,od);
```

we get

```
Number of orientation points = 2
```



6 How to install and operate

1. Download the file
   http://interval.ict.nsc.ru/Programing/MCodes/IntLinInc2D.zip
2. Unpack it into a separate directory.
3. Set MATLAB path to this directory.
4. In the MATLAB command window, input data for systems (1)–(8) and call the functions of the package as shown in the examples of this manual.

7 References

[1] I.A. Sharaya, Boundary interval method and visualization of polyhedral sets, *to appear in Reliable Computing*.

[2] I.A. Sharaya, Quantifier-free descriptions for interval-quantifier linear systems, *Trudy Instituta Matematiki i Mekhaniki UrO RAN [Proceedings of the Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences]*, 20 (2014), No. 2, pp. 311–323. (In Russian)
http://interval.ict.nsc.ru/sharaya/Papers/trIMM14.pdf

[3] S.P. Shary, A new technique in systems analysis under interval uncertainty and ambiguity, *Reliable Computing*, 8 (2002), No. 5, pp. 321–418.
http://interval.ict.nsc.ru/shary/Papers/ANewTech.pdf

[4] J. Rohn, Solvability of systems of interval linear equations and inequalities. In: *Linear optimization problems with inexact data*, M. Fiedler, J. Nedoma, J. Ramik, J. Rohn, K. Zimmermann. New York, Springer, 2006. P. 35–77.
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