NON MODAL GROWTH OF DISTURBANCES IN FREE–SURFACE FLOWS

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Возникновение кратковременного роста малых возмущений в плоских струях жидкости, падающих под действием гравитации в неподвижном газе, исследуется с помощью решения соответствующей задачи с начальными условиями, которая формулируется принимая во внимание эффект вязкости жидкости, поверхностное натяжение и инерцию. Исследуется численный аналог дифференциального оператора, полученный с помощью спектральной дискретизации. Внимание фокусируется и на области числовых значений и на псевдоспектре с целью подчеркивания их характера, отличного от нормального. Также вычисляется направление в зависимости от времени функции роста энергии возмущений. Работа в основном касается физических механизмов, приводящих к появлению такого кратковременного роста энергии возмущений, и показывает, что хотя основное течение является несдвиговым, рост энергии возмущений связан с так называемыми формами поверхности.

Introduction

Recent developments in linear stability analysis of fluid flows have highlighted the possibility for transient growth of the perturbations energy even in situations where the classical normal modes analysis predicts stability [14]. Eigenvalues only depict the response of a basic flow to a perturbation in the far field, and are not able to give a complete solution to the stability problem in cases described by non–normal operators. The non–orthogonality of the eigenfunctions may lead, in these situations, to amplification of inputs to the system by arbitrarily large factors over short time or space scales.

These arguments opened the way to explain transition in Poiseuille or Couette flows [9][10], and later on in boundary layers [1], as due to short time linear phenomena linked to such transient effects rather than to the exponential growth in time or space. In a recent paper de Luca et al. [4] considered the effects of non–normality in fluid flows with a free interface. The instability of a plane liquid jet issuing in a still atmosphere was taken under examination and the evolution equations describing the initial value problem for the perturbations were written within a viscous model. Discretization by means of a spectral technique allowed the evaluation of the operator pseudospectra and numerical range together with a proper growth function for the perturbations energy. The control parameter of this kind of problem is the local Weber number, expressing the relative influence of inertia to surface tension forces. Previous works of literature [3][5][7][8][11][12] stated that the critical value of this parameter, below which modal instabilities occur, is the unity, which is practically independent of the jet Reynolds number and the gas–to–liquid density ratio. de Luca et al. [4] investigated the non–normality of the governing operator and found transient energy growths of perturbations for ranges of Weber values where the standard eigenvalues Lyapunov's analysis predicts an asymptotic stability in time. As a consequence they argued that the jet break–up experimentally observed in decreasing the liquid flow rate to be caused by such a transient growth, which is eventually followed by non–linear interactions.

The present paper draws his motivation from the above described new conception of hydrodynamic stability problems and extends ideas and results about the physical mechanisms that are responsible for the transient growth of disturbances. Indeed, it is shown that, in spite of the fact that the base flow is unsheared, the so called surface modes determine such a non–modal amplification of disturbances energy. This conclusion is corroborated by a detailed analysis of the numerical range of the governing operator.

1. The Initial Value Problem

The model refers to a liquid falling under gravity from a thin slit into a gas atmosphere. Following the paper of de Luca et al. [4] to which the reader is referred to go into details, linear theory of stability is applied by superimposing small amplitude perturbations to the basic state. This last is characterised by a velocity of the internal fluid dominated by inertia and gravity forces. The pressure p and the velocity components u, v and w, referred to a co–ordinate system having the x–axis in the spanwise direction and y and z as the lateral and the vertical (or streamwise) directions, are defined by introducing the following dimensionless quantities:

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$$x = \frac{x^*}{b} \qquad y = \frac{y^*}{b} \qquad z = \frac{z^*}{b} \qquad t = \frac{t^* \overline{U}}{b}$$

$$p_j = \frac{p_j^*}{\rho_j \overline{U}^2} \quad u_j = \frac{u_j^*}{\overline{U}} \quad v_j = \frac{v_j^*}{\overline{U}} \quad w_j = \frac{w_j^*}{\overline{U}}$$

with j=1, 2, where subscripts 1 and 2 refer to the internal (liquid) and external (air) motion, respectively. The time variable is denoted by t. $\overline{U}$ is the mean vertical velocity at the slit exit section, b is the slit half–width and asterisks refer to dimensional variables. The interface position η is also made dimensionless with respect to b. Assuming ρ_j as the density and μ_j as the viscosity of the two fluids and σ as the liquid surface tension, the Reynolds numbers:

$$\mathbf{Re}_j = \frac{\rho_j \overline{U} b}{\mu_j}, \tag{1}$$

the Weber number

$$We = \frac{\rho_1 \overline{U}^2 b}{\sigma}, \tag{2}$$

and the density ratio

$$r = \frac{\rho_2}{\rho_1} \tag{3}$$

are also introduced. The hypotheses are made that both the motions are incompressible and that the viscosity of the external fluid is negligible with respect to the internal one. Re_2 is thus infinite and Re_1, the liquid Reynolds number, will be later on denoted simply by Re. Navier–Stokes equations are written for the perturbed velocity field together with boundary conditions on the interface, the symmetry axis for the internal region and the far field for the external one. Sinuous (symmetric) perturbations are only considered which displace both the free surfaces in the same direction.

Perturbation quantities, ψ'_j, are defined as given by:

$$\psi'_j(x,y,z,t) = \psi'_j(x,y,Z,t:\delta) = \mathbf{exp}\left(i\beta x + i\frac{\theta(Z)}{\delta}\right)\sum_{k=0}^{\infty}\delta^k \psi_{kj}\left(y,Z,t:\delta\right), \; j=1,2. \tag{4}$$

Note that while Fourier components are taken along the x–direction, based on the invariance of the main flow, the time dependence is retained. As regards the z–direction, multiple scales, amenable to be introduced on the ground of the hypothesis of slow streamwise variations of the basic flow, are based on the small parameter

$$\delta = \frac{2gb}{\overline{U}^2}, \tag{5}$$

where g is the gravity acceleration. Z and z have to be considered as the slow and fast length scales, respectively, related to each other by $Z=\delta z$, as proposed by de Luca and Costa [3].

A convenient formulation of the problem can be done in terms of the y–component of velocity and vorticity:

$$\omega_{01} = i\alpha u_{01} - i\beta w_{01}, \tag{6}$$

by introducing the streamwise wavenumber:

$$\alpha = \frac{d\theta}{dZ}. \tag{7}$$

The governing system of equations, written at the zeroth order approximation, may be re–arranged as shown in the following, with $\kappa^2 = \alpha^2 + \beta^2$:

$$\frac{\partial \omega_{01}}{\partial t} = \frac{1}{\mathbf{Re}}\left(\frac{\partial^2}{\partial y^2} - \kappa^2\right)\omega_{01} - i\alpha\omega_{01}, \tag{8.a}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial y^2} - \kappa^2\right)v_{01} = -i\alpha\left(\frac{\partial^2}{\partial y^2} - \kappa^2\right)v_{01} + \frac{1}{\mathbf{Re}}\left[\frac{\partial^4}{\partial y^4} - 2\kappa^2\frac{\partial^2}{\partial y^2} + \kappa^4\right]v_{01}, \tag{8.b}$$

$$\left[r+\mathbf{tanh}(\kappa)\right]\frac{\partial v_{02}}{\partial t}=-\mathbf{tanh}(\kappa)\left[i\alpha v_{02}+\left(i\alpha+\frac{2\kappa^2}{\mathbf{Re}}\right)v_{01}\right]-\frac{2\kappa}{\mathbf{Re}}\frac{\partial v_{01}}{\partial y}-\frac{\kappa^3\eta_0}{We}\qquad y=1 \tag{8.c}$$

$$\frac{\partial \eta_0}{\partial t}=v_{02}\qquad y=1 \tag{8.d}$$

with boundary conditions:

$$\omega_{01}=0\qquad y=0 \tag{9.a}$$

$$\frac{\partial \omega_{01}}{\partial y}=0\qquad y=1 \tag{9.b}$$

$$\frac{\partial v_{01}}{\partial y}=0\qquad y=0 \tag{9.c}$$

$$\frac{\partial^3 v_{01}}{\partial y^3}=0\qquad y=0 \tag{9.d}$$

$$\frac{\partial^2 \mathrm{v}_{01}}{\partial \mathrm{y}^2}+\kappa^2 v_{01}=0\qquad y=1 \tag{9.e}$$

$$v_{01}-i\alpha\eta_0=v_{02}\qquad y=1 \tag{9.f}$$

Note that in eqs. (8) and (9) the dimensionless variables are re–defined on a local basis substituting b by the local half–thickness $\overline{\eta}_0$ and the mean velocity $\overline{U}$ by the local velocity $\overline{w}_{01}$. $\overline{\eta}_0$ and $\overline{w}_{01}$ represent the leading terms of a power series expansion in δ.

Consequently the Weber number is hereafter re–defined as the local parameter

$$We=\frac{\rho_1\overline{w}_{01}^{2}\overline{\eta}_0}{\sigma}, \tag{10}$$

while the definition of Re remains unchanged.

Eqs. (8.a) and (8.b), to be integrated on the domain $0\le y\le 1$, represent the time evolution equations for ω_{01} and v_{01}. Eqs. (8.c) and (8.d), arising as boundary conditions apparently in excess, but really involving the two additional unknowns v_{02} and η_0, which are functions of time only (besides the slow co–ordinate Z), are considered as evolution equations as well. Furthermore the initial–boundary value problem for ω_{01} is not coupled with that for v_{01} and can thus be solved independently.

In summary, the linear evolution equations (8) can be written as the initial value problem:

$$S\frac{\partial q}{\partial t}=Tq \tag{11}$$

with some given initial conditions $q(y,0)$, S and T as two differential operators, and q as the vector of unknowns:

$$q(y,t)=\begin{bmatrix}\omega_{01}(y,t)\\ v_{01}(y,t)\\ v_{02}(t)\\ \eta_0(t)\end{bmatrix}. \tag{12}$$

2. Numerical Discretization and Computational Procedure

A spectral algorithm based upon collocation has been developed. We used N–th order Chebyshev polynomials $T_N(\xi)$ defined on the domain $-1\le\xi\le 1$, which corresponds to the domain $0\le y\le 1$ via the co–ordinate transformation:

$$\xi=2y-1. \tag{13}$$

Variables ω_{01} and v_{01}, are written in the form:

$$\omega_{01}(\xi,t)=\sum_{k=0}^{N}a_k(\xi)\hat{\omega}_{01}(\xi_k,t), \tag{14}$$

$$v_{01}(\xi,t)=\sum_{k=0}^{N} a_k(\xi)\hat{v}_{01}(\xi_k,t), \tag{15}$$

where $\hat{\omega}_{01}$ and $\hat{v}_{01}$ are two *(N+1)*–th order vectors, representing values of the two unknowns over the collocation points:

$$\xi_k=\cos\frac{\pi k}{N},\ \ k=0,1,\dots,N, \tag{16}$$

and the interpolant for the Chebyshev scheme is given by:

$$a_k(\xi)=\left(\frac{1-\xi_k^2}{\xi-\xi_k}\right)\frac{T_N'(\xi)}{N^2c_k}(-1)^{k+1}, \tag{17}$$

with $c_0=c_N=2$, and $c_k=1$ for k=1,2,…N–1.

Equations (8.a) and (8.b) are written at the collocation points. According to Canuto et al. [2], and following the scheme developed by de Luca et al. [4], Neumann boundary conditions, described by eqs. (9.b–e), are treated indirectly by imposing them in the construction of numerical derivative matrices within a cascade procedure.

By adding eqs. (8.c) and (8.d) to the *2(N+1)* equations for $\hat{\omega}_{01}$ and $\hat{v}_{01}$, obtained by enforcing eqs. (8.a) and (8.b) on the grid points, the *(2N+4)*–th order linear system is obtained:

$$\hat{S}\frac{d\hat{q}}{dt}=\hat{T}\hat{q}, \tag{18}$$

which is the discrete counterpart of the initial value problem (11). The state vector is:

$$\hat{q}=\left[\hat{\omega}_{01}\quad \hat{v}_{01}\quad v_{02}\quad \eta_0\right]^T. \tag{19}$$

The boundary condition (9.f) is imposed by dropping the differential equation (8.b) at the collocation point k=0 (corresponding to the boundary y=1) and by using eq. (9.f) to eliminate the unknown liquid velocity $\hat{v}_{01}$ at k=0.

The classical *QZ*–algorithm solves the generalised eigenvalue problem for eq. (18). Remember that the equation for vorticity, with its boundary conditions, is decoupled from others and its spectrum derived independently.

3. Stability Analysis in Energetic Hilbert Space

To our purposes it is convenient to introduce the energy norm as the governing norm for evaluating the disturbances temporal evolution. To do so let us introduce the non–dimensional disturbance energy:

$$E(t)=\frac{1}{2\kappa^2}\left[\int_0^1\left(\left|\frac{\partial v_{01}}{\partial y}\right|^2+\kappa^2|v_{01}|^2+|\omega_{01}|^2\right)dy+\frac{1}{We}\kappa^4|\eta_0|^2\right], \tag{20}$$

which represents a measure of the perturbations kinetic energy and the potential elastic energy due to surface tension. This energy, which is adimensionalised with respect to the reference energy $\rho\overline{U}^2b$, is considered in the wave number space. Accordingly, in order to obtain the equivalent energy in the physical space, it is necessary to integrate all over the streamwise and spanwise wavenumbers.

With the aid of such an energy it is possible to define the energetic growth function in the phase space $\Xi=(\alpha,\beta,\mathrm{Re},We,r)$:

$$G(t;\Xi)=\sup_{E(0)\neq 0}\frac{E(t)}{E(0)}, \tag{21}$$

which yields, at any time t, the ratio of the system perturbations energy with respect to its initial value.

Note that a growth function can be equivalently defined upon considering the following Cauchy's problem for some state vector f:

$$\frac{d}{dt}f(t)=Lf(t),\ \ f(0)=f_0, \tag{22}$$

the solution of which is formally $f(t)=f_0e^{Lt}$. The corresponding growth function is:

$$G_\phi(t)=\frac{\|f\|_\phi}{\|f_0\|_\phi}=\left\|e^{Lt}\right\|_\phi. \tag{23}$$

Now, if the norm ϕ is the energy one (from now onwards we will put ϕ=H, where H stands for energy norm), the function G_H and the function $G(t;\Xi)$ are the same, e.g. the growth function can be equivalently written as the H–norm of the exponential of Lt.

A convenient procedure for evaluating the H–norm in eq. (23) consists in performing an eigenfunction expansion of operator L in order to restrict attention to the first (more significant) k modes; such an operation advantageously allows to switch from an infinite dimensional space to a k–dimensional space.

According to Reddy, Schmid and Henningson [10] the growth function of eq. (23) can be written in the eigenfunctions basis as:

$$G_K(t) = \left\| F \Lambda F^{-1} \right\|_2^2, \tag{24}$$

where

$$\Lambda = diag\left(e^{\lambda_1 t}, \ldots, e^{\lambda_K t}\right) \tag{25}$$

is the (unsteady) matrix built with the first k eigenvalues and the (steady) matrix F is obtained by the Cholesky's decomposition of the matrix Q, $Q = F^* F$ (star denoting the Hermitian conjugate). Matrix Q is obtained as:

$$Q_{ji} = \left\langle D\hat{v}_{01}^j, D\hat{v}_{01}^i \right\rangle_L + \kappa^2 \left\langle \hat{v}_{01}^j, \hat{v}_{01}^i \right\rangle_L + \left\langle \hat{\omega}_{01}^j, \hat{\omega}_{01}^i \right\rangle_L + \frac{\kappa^4}{We} \eta_0^{i\,*} \eta_0^j, \tag{26}$$

where the j–th and i–th eigenfunctions are taken in pair and the inner product $\langle \cdot, \cdot \rangle_L$ is defined as:

$$\left\langle \hat{v}_{01}^j, \hat{v}_{01}^i \right\rangle_L = \int_0^1 \hat{v}_{01}^{i\,*} \hat{v}_{01}^j dy. \tag{27}$$

The 2–norm in eq. (24) can be determined e.g. by performing a singular value decomposition (SVD) of the matrix $F \Lambda F^{-1}$.

Another paramount aspect of the non–normality of L can be outlined by its energetic numerical range defined as

$$W_H(L) = \left\{ f^* L f : \|f\|_H = 1 \right\}, \tag{28}$$

i.e. by the subset of the complex plane given by the energetic Hilbert scalar product $\langle Lv, v \rangle_H$ performed with all vectors in C^N having energetic norm equal to the unity.

A fundamental consequence of Hille–Yosida's theorem [13] consists in relating the numerical abscissa $\omega(L)$, defined as max Re$\{W_H(L)\}$, to the growth function:

$$\omega(L)\big|_{\|\bullet\|_H} = \frac{d}{dt} \left\| e^{Lt} \right\|_H. \tag{29}$$

From eq. (29) it easy to infer that an asymptotically stable operator, i.e. having a negative spectral abscissa $\alpha(L)$:

$$\alpha(L) = \max_i \mathbf{Re}(\lambda_i), \tag{30}$$

but a positive numerical abscissa has a non–monotone decaying (energetic) growth function. The closure of the numerical range of a normal operator is the convex hull of its spectrum [6].

Closely linked to the growth function and numerical range analysis performed in the energy norm, is the ε–pseudospectral analysis. It is known that the response of a system to a perturbation does not depend only on the eigenvalues of the relevant governing operator, namely the frequencies where the norm of the resolvent of the initial value problem is infinite, but, in the case of non–normal operators, it depends also on frequencies loci where this norm is large enough. For a normal operator the ε–pseudospectrum is simply the union of the closed disks of radius ε centred at eigenvalues. For non–normal operators the ε–pseudospectrum can be much larger, even if $\varepsilon \ll 1$. For more general aspects the reader is referred to Trefethen [13].

In energetic norm ε–pseudospectra are represented by the following subsets of the complex plane:

$$\Lambda_\varepsilon(L) = \left\{ z \in C : \left\| (zI - L)^{-1} \right\|_H \geq \varepsilon^{-1} \right\}. \tag{31}$$

Upon performing again an eigenfunction expansion of the resolvent of operator L, given by $R = (zI - L)^{-1}$, it is possible to calculate the energetic norm of the resolvent and thus the energetic ε–pseudospectra as:

$$\left\| (zI - L)^{-1} \right\|_H \approx \left\| \left(zI - F \Lambda F^{-1} \right)^{-1} \right\|_2. \tag{32}$$

4. Results

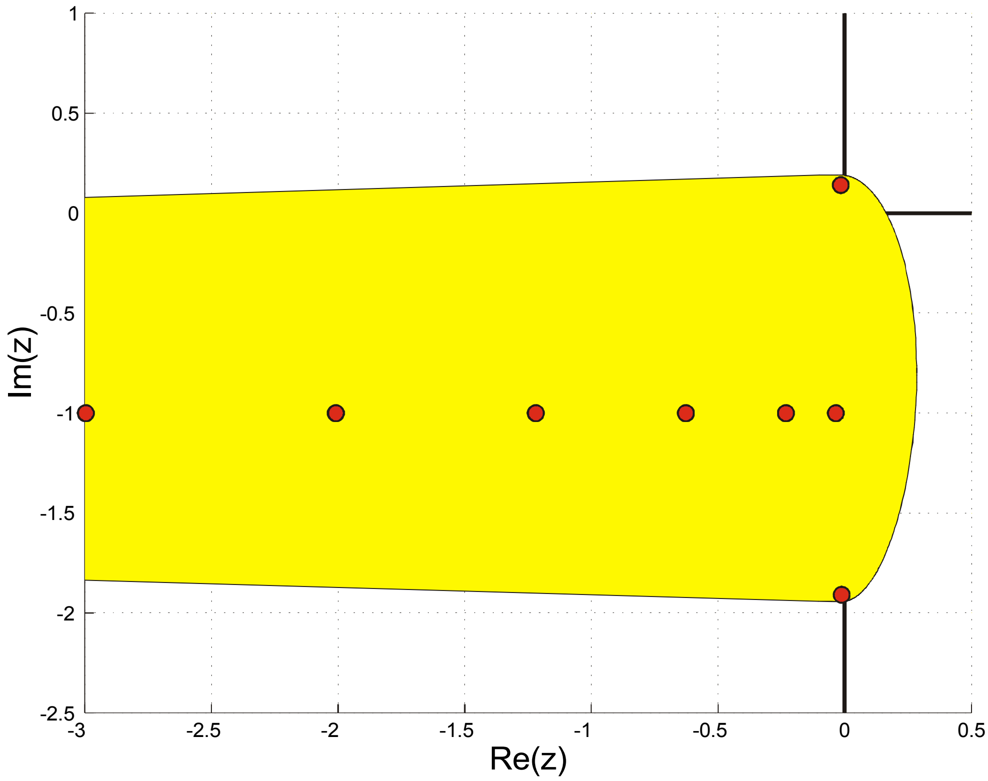

Fig.1: Numerical range computed including stretch axis and surface modes.

Fig. 1 shows a typical numerical range referring to the following parameters: Re=100, We=1, r=0.1, α=1, β=0. Computation has been performed choosing N=60, and restricting attention on the first k=20 least stable modes. From one hand this reduces the computational effort, on the other one allows one to work by taking into account the accurately computed eigenvalues and the corresponding eigenfunctions only. As shown in Fig. 1 the numerical range of the discretised counterpart of the differential governing operator is a semi–infinite strip with two edges (almost) parallel to the real axis. The fact that the horizontal edges do not appear rigorously parallel to the real axis is a numerical effect and clearly depends on the finite number k of eigenfunctions included into the computation. In fact, increasing k increases the number of eigenvalues computed numerically which extends the spectrum into the negative real half plane. As k tends to infinity the exact spectrum is recovered which extends to infinity on the left side, thus producing a numerical range with two edges rigorously parallel to the real axis.

The right edge protrudes into the positive complex half–plane, thus, although the test case here chosen refers to a asymptotic decay of disturbances (i.e., the system is stable following Lyapunov), on the grounds of the theorem of Hille–Yosida, the presence of transient growth of disturbances is to be expected. Note also that for the present case the numerical abscissa is a relatively small positive quantity (ω≈.28), accordingly one may expect a relatively small amount of the energy growth function. This will be discussed in more details later on.

The solid circles reported within the numerical range denote the location of eigenvalues. Remember that the equation (8.a) of the vorticity is not coupled with the equation (8.b) of the velocity, together with the proper boundary conditions. As discussed in a detailed way by de Luca, Costa and Caramiello [4], velocity eigenvalues belong to two families, one grouping those with imaginary part equal to $-\alpha$, the other one containing two eigenvalues which will be hereafter denoted off–axis eigenvalues. The former group of velocity eigenvalues coincide practically (but not exactly due to the coupling of equation 8.b with equations 8.c and 8.d) with vorticity eigenvalues, the corresponding eigenfunctions being almost orthogonal to each other. The almost "normal" character of these stretch modes is evidenced by the plot of Fig. 2, where the numerical range of the matrix operator containing the stretch modes only is reported. The matrix being very weakly non–normal, the numerical range practically reduces to the segment spanning the stretch eigenvalues, and thus all lying within the negative real part of the complex plane. No disturbances energy transient growth is expected in this case.

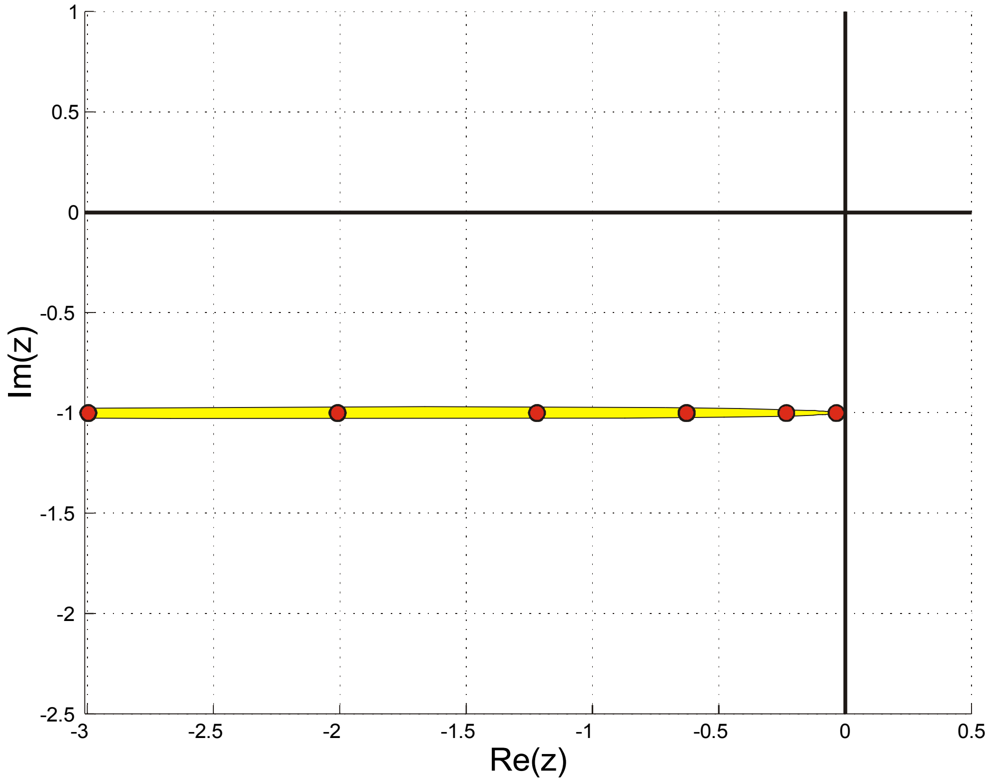

Fig.2: Numerical range computed including stretch axis modes only.

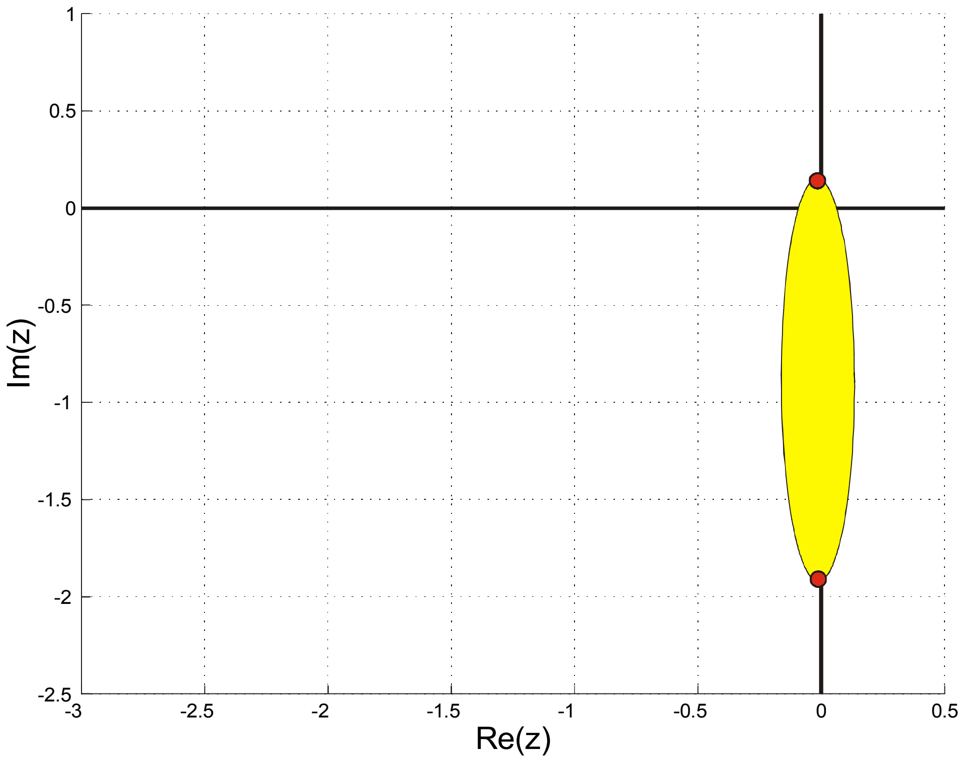

Fig.3: Numerical range computed including the two surface modes only.

As a consequence, the potential for a transient growth should be attributed solely to the off–axis modes. Fig. 3 plots the numerical range of the 2x2 matrix corresponding to the two off–axis eigenvalues. These are clearly linked to the evolution equations (8.c) and (8.d), which express free–surface boundary conditions. For this reason they are named also "surface" modes. The numerical range of Fig. 3 denotes a non–normal character of the matrix, and protrudes into the positive real half–plane. It should be stressed that, as a general property of 2x2 matrices, the numerical range is an ellipse whose foci are the eigenvalues of the matrix (Gustafson and Rao [6]). Furthermore, since the non–diagonal terms of the matrix are very small as compared to the diagonal ones, the foci are almost located next to the ellipse boundary.

As a summary, from the analysis of the matrix operator carried out above it may be argued that its non–normal character as well the potential for transient growth of energy disturbances are clearly to be attributed to the "surface" modes. In other words, contrary to all previous studies of transient growth in fluids, e.g. Couette and Poiseuille flows, where the shear is responsible for the so–called "lift–up" mechanism, here the base flow does not contain any shear; nevertheless, a transient growth of energy disturbances does occur, due to the afore mentioned "surface" modes.

Figs. 4, 5, and 6 depict the time evolution of the energy of disturbances for the cases corresponding to Figs. 1, 2, and 3, respectively. Thus, when all (computed) modes are included, the typical trend of Fig. 4 is obtained. Note that the relatively small amount of the maximum value attained by the growth function has to be related to the relatively small value of the numerical abscissa shown in Fig. 1. It has to be stressed that the continued oscillations of the growth function should be due to the circumstance that the matrix operator has two eigenvalues of practically coincident real parts (Trefethen [13]). In the case that the stretch modes only are taken into account (Fig. 5) an exponential decay is found, in agreement with the previous finding of Fig. 2, where it has been shown that the numerical range is completely confined within the negative real half–plane. Finally, Fig. 6 shows that the two surface modes alone are sufficient to produce a transient growth of disturbances. Numerical results, not shown in this paper, show also that such a growth is also obtained when one considers just one surface mode coupled with any stretch mode.

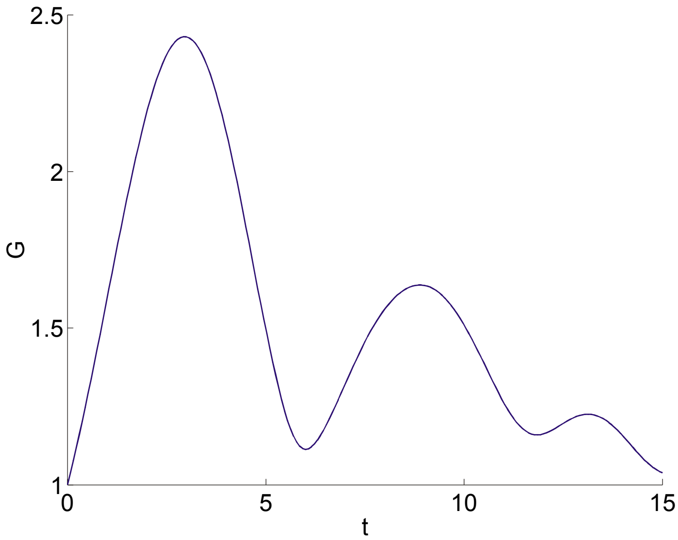

Fig.4: Growth function computed including stretch axis and surface modes.

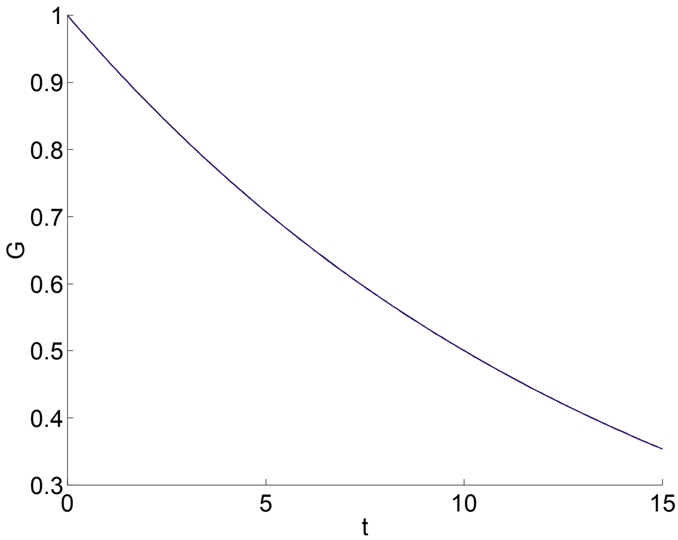

Fig.5: Growth function computed including stretch axis modes only.

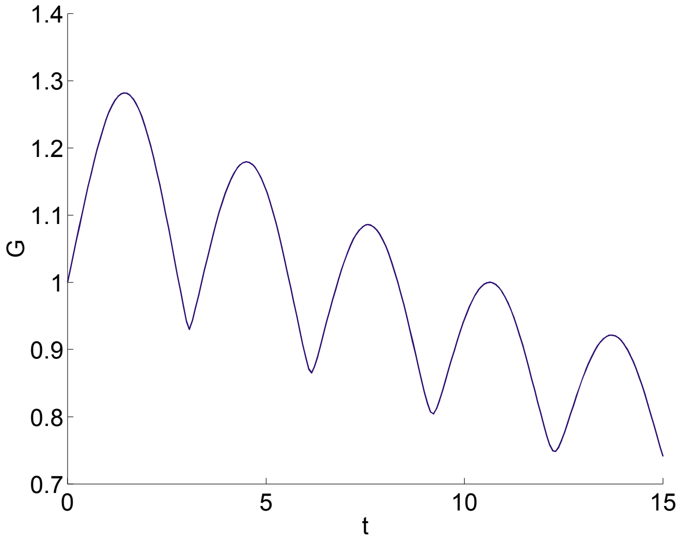

Fig.6: Growth function computed including the two surface modes only.

Finally, pseudospectra iso–level curves computed for the same physical case as previous figures is plotted in Fig. 7. As usual, the numerical values reported on the different curves correspond to the amplification ratio ε of forced output to forcing input amplitudes. If the operator would be normal a ε–level curve should be a circumference of radius ε^{-1} centered at a certain eigenvalue. If the operator is not normal a remarkable amplification remains at a radius greater that ε^{-1}. From an analysis of Fig. 7 it arises that present operator is slightly non–normal; this result agrees with the analysis of the numerical range previously reported in Fig.3. It should be remembered that the non–normality is just a necessary condition, but not sufficient in order to have a transient growth for which it is necessary that the numerical range protrudes into the positive real half–plane. The present pseudospectra lines are computed over a 150x150 grid covering the complex plane.

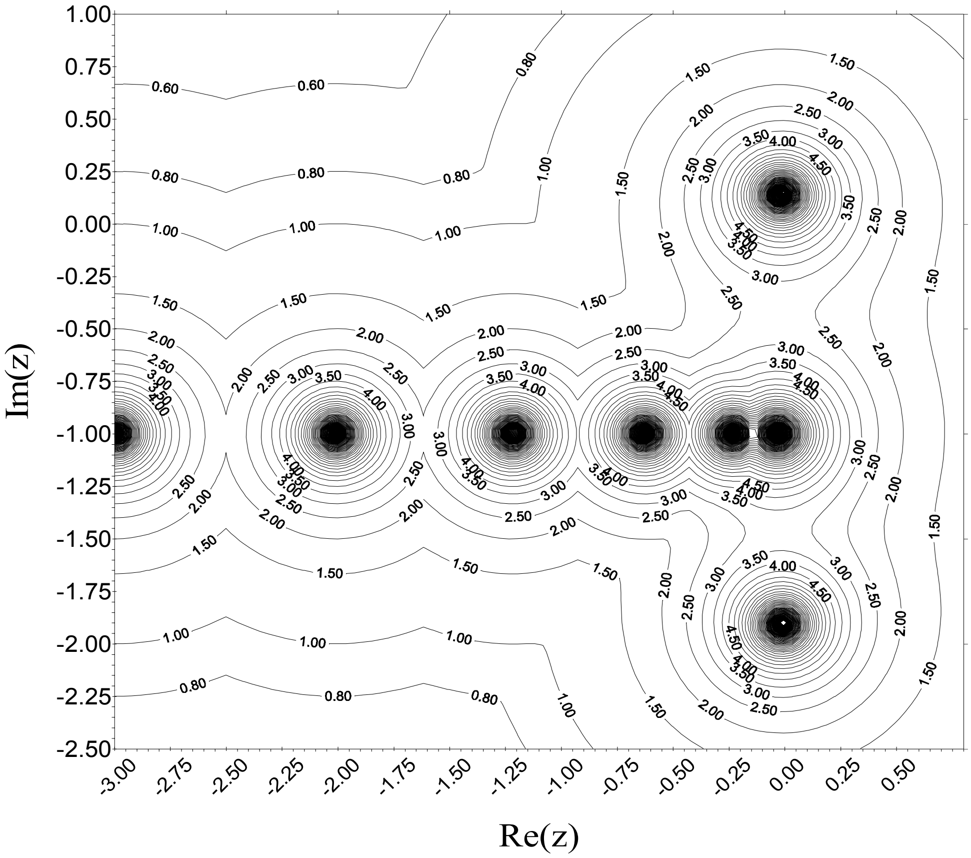

Fig.7: ε–pseudospectra iso–level curves.

Conclusions

The instability of a plane liquid jet issuing vertically from a long horizontal slit into a still atmosphere is analysed and the evolution equations describing the initial value problem for the perturbations are written within a model including viscous, surface tension and inertia effects.

The present paper draws its motivation from a relatively new conception of hydrodynamic stability problems, which highlights the possibility for transient growth of the perturbations energy even in situations where the classical normal modes analysis predicts stability. Eigenvalues only depict the response of a basic flow to a perturbation in the far field, and are not amenable to give a complete solution of the stability problem in cases described by non–normal operators. The non–orthogonality of the eigenfunctions may lead, in these situations, to amplification of inputs to the system by arbitrarily large factors over short time or space scales. Indeed, the non–normality is just a necessary condition, because in order to have such energy disturbances transient growths the relevant numerical range must protrude into the positive real half–plane.

In a previous paper present authors already studied the effects of non–normality in fluid flows with a free interface, but restricted their analysis to computations of pseudospectra curves as well as time trends of the disturbances growth function. The present work extends ideas and results about the effects of non–normality on liquid jets in air examining the characteristics of the numerical range of the relevant matrix operator, which is the discretised counterpart of the governing differential operator.

It is shown that the numerical range protrudes into the positive real half–plane basically due to the presence of the so–called "surface modes", i.e. the eigenfunctions corresponding to the eigenvalues strictly linked to the two free surface boundary conditions enforcing the normal stress jump due to the surface tension and the kinematic congruence. Computations show that the numerical abscissa is relatively small, thus producing a corresponding relatively small level of the short time growth of the disturbances energy. Iso–level curves of pseudospectra also denote a relatively small non–normal character of the operator.

As a final physical consideration, from the analysis of the matrix operator carried out in this paper it may be argued that its non–normal character as well as the potential for transient growth of energy disturbances are clearly to be attributed to the "surface" modes. In other words, contrary to all previous studies of transient growth in fluids, e.g. Couette and Poiseuille flows, where the shear is responsible for the so–called "lift–up" mechanism, here the base flow does not contain any shear; nevertheless, a transient growth of energy disturbances does occur, due to the afore mentioned "surface" modes.

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